Summary

Groups acting on trees

June 5, 2021

This summary is not the complete content of the class, statements are less detailed than in the notes, and the proofs of some results are just as important as the statements themselves. So please use this only as a guide! The numbering of the sections follows that of the lectures, and some results from the exercise sets are incorporated.

1 Graphs and automorphisms of trees

Definition 1.1. A *graph* is a pair $X = (X^0, X^1)$ where X^0 is the set of *vertices* and X^1 is the set of *edges*. This comes with functions $\alpha, \omega : X^1 \to X^0$ giving the source and target of an edge, and a fixpoint-free involution $\bar{\cdot} : X^1 \to X^1$ giving the reverse edge. An *orientation* is a subset $X^1_+ \subset X^1$ where we choose one of e or $\overline{e}$ for every edge.

Definition 1.2. A *path* is a finite sequence of consecutive edges, i.e, the target of an edge is the source of the next edge. It is *reduced* if an edge is never followed by its reverse. A graph is *connected* if any two vertices can be joined by a path. The *distance* between two vertices is the minimal length of a path joining them.

Definition 1.3. A *tree* is a connected graph containing no circuits, i.e., no reduced path starting and ending at the same vertex.

Remark. In a tree, any reduced path achieves the distance between its two endpoints.

Definition 1.4. An *automorphism* of a graph X is an invertible map τ sending vertices to vertices and edges to edges, which preserves the structure (namely the maps $\alpha, \omega, \bar{\cdot}$). It acts *without inversion* if $\tau(e) \neq \overline{e}$ for every edge.

From now on, we assume that all automorphisms act without inversion. For the rest of this section, X is a tree and $\tau \in Aut(X)$.

Definition 1.5. Define $|\tau|$ to be the minimum of $d(v, \tau(v))$ over all vertices. This is called the *translation length.* If $|\tau| = 0$, we call it a *rotation*; else a *translation.*

Theorem 1.6. *Suppose that τ is a rotation, and let $\mathring{\tau}$ be the set of vertices and edges which are fixed by τ. Then $\mathring{\tau}$ is a (non-empty) tree.*

Suppose that τ is a translation. Then the vertices where the minimum is attained lie on a bi-infinite line $\vec{\tau}$ where τ acts as a translation by $|\tau|$.

Applying this theorem and a few more things, we get:

Corollary 1.7. *Any finite group of tree automorphisms has a global fixpoint.*

2 Letting groups act on graphs

Definition 2.1. An *action* of a group G on a graph X is a homomorphism $G \to Aut(X)$; we denote the action simply by $x \mapsto gx$ or $g \cdot x$. The stabilizer of $x \in X^0 \cup X^1$ is denoted by G_x: note that the stabilizer of an edge is contained in the stabilizer of its two endpoints. The action is *free* if all stabilizer are trivial. The action is *without inversion* if each of the corresponding automorphisms is.

Remark. An action on X can be made without inversion by passing to the *barycentric subdivision*. So we will always assume that this is the case.

Finitely generated groups always have meaningful actions on graphs.

Definition 2.2. Let G be a group with a finite generating set S. The corresponding *Cayley graph* is the graph $\Gamma(G, S)$ with vertex set G and positively oriented edges of the form $g \to gs$ for $g \in G, s \in S$. It is connected and G acts on it freely and without inversion.

Definition 2.3. Given an action of G on a graph X, we can form the *quotient* $G\backslash X$, which is the quotient graph given by identifying orbits of vertices and edges. This comes equipped with the natural *projection* $p : X \to G\backslash X$.

Proposition 2.4. *If $T \subset G\backslash X$ is a tree, then it admits a* lift*, i.e., there exists a tree $\tilde{T} \subset X$ such that p maps $\tilde{T}$ isomorphically onto T.*

3 Elementary properties of free groups and presentations

Definition 3.1. A group F is *free* with *basis* $X \subset F$ if any element of F can be written uniquely as a reduced product of elements in $X \sqcup X^{-1}$ (i.e., x is never followed by x^{-1}). We denote F also by $F(X)$.

Proposition 3.2. *$F(X)$ has the following properties (for $|X| > 1$):*

1. *It is determined up to isomorphism by $|X|$, which is called the* rank *of $F(X)$;*
2. *Its Cayley graph with respect to X is a tree (this even characterizes $F(X)$);*
3. *It is torsion-free;*
4. *It has trivial center;*
5. *It contains free subgroups of finite index and arbitrarily large finite rank;*
6. *Its abelianization is the free abelian group $\mathbb{Z}[X]$.*

The fundamental property of free groups is the following:

Theorem 3.3. *$F(X)$ has the following universal property: for every group G, any map $X \to G$ extends uniquely to a homomorphism $F(X) \to G$.*

Corollary 3.4. *Any group is isomorphic to a quotient of a free group.*

Explicitly, if G is generated by $S \subset G$, and X is an abstract set in bijection with S, the map $X \cong S \subset G$ extends to a surjective homomorphism $F(X) \to G$. Let N be the kernel of this map, so that $G \cong F(X)/N$.

Definition 3.5. If $R \subset N$ is a normal generator of N (namely, N is the smallest normal subgroup of $F(X)$ containing R, or the subgroup generated by R and all of its conjugates in $F(X)$), then we write $G = \langle X \mid R \rangle$ or $\langle X \mid R = 1 \rangle$ and call this a *presentation* of G. It determines G up to isomorphism.

Example 3.6. Here are some presentations:

1. $F(X) = \langle X \mid \rangle$.
2. $\mathbb{Z}/n\mathbb{Z} = \langle x \mid x^n = 1 \rangle$.
3. $\mathbb{Z}[X] = \langle X \mid [x, y] = 1 : x, y \in X \rangle$.
4. $G \times H = \langle G, H \mid [g, h] = 1 : g \in G, h \in H \rangle$.
5. $G \rtimes_\varphi H = \langle G, H \mid hgh^{-1} = \varphi(h)(g) : g \in G, h \in H \rangle$.

In the last two presentation, the notation $\langle$groups $\mid$ relations$\rangle$ means that we take the generators of the groups to be disjoint sets, and are omitting from the notation the relations defining the groups. Note also that we wrote $g \in G, h \in H$, but including these relations only for a set of generators is enough. In particular, if G, H are finitely presented, so is their (semi)direct product.

Presentations make it easy to define homomorphisms:

Theorem 3.7. *Let $G = \langle X \mid R \rangle$, and consider a group H and a map $X \to H$. If the corresponding homomorphism $F(X) \to H$ sends R to 1, then it induces a unique homomorphism $G \to H$.*

4 Free groups and graphs

Definition 4.1. Let X be a connected graph and $x \in X^0$. Two paths are *homotopic* if they coincide after reduction (i.e., deleting consecutive occurrences of e and $\overline{e}$). The *fundamental group* $\pi_1(X, x)$ is the group of homotopy classes of paths starting and ending at x, with multiplication given by concatenation.

Theorem 4.2. *The fundamental group of a graph is free.*

An explicit basis can be given, which depends on the choice of a *spanning tree*, i.e., a tree $T \subset X$ such that $T^0 = X^0$. Developing on this, we obtain:

Theorem 4.3. *Let G be a group acting freely and without inversion on a tree. Then G is free.*

The converse also holds since a free group acts freely and without inversion on its Cayley graph, which is a tree.

Corollary 4.4 (Nielsen–Schreier)**.** *Every subgroup of a free group is free. If G is free of finite rank and $H \leq G$ is of finite index, then*

$$[G : H] = \frac{rk(H) - 1}{rk(G) - 1}.$$

The formula in the previous corollary comes from the more precise versions of the previous theorems.

5 Free products and playing ping-pong

Definition 5.1. Let A and B be two groups. Their *free product* is the group $A * B = \langle A, B \mid \rangle$. Every element can be written uniquely as an alternating product of elements of A and of B: such an expression is called a *normal form.*

Theorem 5.2. *The existence and uniqueness of normal forms characterizes the free product. Namely if G contains two subgroups A, B such that every element of G can be written uniquely as an alternating product of elements of A and of B, then $G \cong A * B$.*

There will be versions of this theorem later on for other constructions, and we will omit the explanation of what "characterizes" means.

Proposition 5.3. *$A * B$ has the following properties (for $|A|, |B| > 1$):*

1. *All its torsion is conjugate to an element of A or B;*
2. *It has trivial center;*
3. *It cannot be written as a non-trivial direct product;*
4. *The kernel of the natural homomorphism $A * B \to A \times B$ is free.*

Here is the main tool for proving that a group is a free product (or a free group):

Lemma 5.4 (Ping-pong lemma)**.** *Let G be a group acting on a set X, $A, B \leq G$ and $X_A, X_B \subset X$ with $X_B \not\subseteq X_A$. Suppose that $g(X_B) \subset X_A$ for all $1 \neq g \in A$, and that $g(X_A) \subset X_B$ for all $1 \neq g \in B$. Then A and B generate a group isomorphic to $A * B$.*

This can be used to prove the following:

Theorem 5.5. $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

Proposition 5.6. *The matrices*

$$x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}; \quad y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generated a free finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

6 Amalgamated products

This is the first of two fundamental constructions in Bass–Serre theory.

Definition 6.1. Let G, H be two groups, $A \leq G, B \leq H$ subgroups and $\varphi : A \to B$ an isomorphism. The *free product of G and H amalgamated over A* is the group $G *_A H = \langle G, H \mid a = \varphi(a) : a \in A \rangle$.

It suffices to add the relation for a set of generators of A, so if G, H are finitely presented and A is finitely generated, then $G *_A H$ is finitely presented. The free product is the special case $A = \{1\}$.

Definition 6.2. Let T_A be a system of representatives of right cosets $A\backslash G$, and T_B of $B\backslash H$, both containing 1. An *A-normal form* for an element $x \in G *_A H$ is a decomposition $x = x_0 x_1 \cdots x_n$ where $x_0 \in A$, and the successive x_i are an alternating product of elements of $T_A \setminus \{1\}$ and $T_B \setminus \{1\}$. One can analogously define a B-normal form.

See Exercise set 3 to practice normal forms on the *torus knot groups*.

Theorem 6.3. *Every element of $G *_A H$ can be written uniquely in A-normal form. Moreover, the existence and uniqueness of A-normal forms characterizes $G *_A H$.*

Corollary 6.4. *G and H embed naturally as subgroups of $G *_A H$ and their intersection is A.*

The first version of the Fundamental Theorem of Bass–Serre Theory describes amalgamated free products as groups acting on trees such that the quotient is a segment (i.e., a graph with two vertices and two edges).

Theorem 6.5. *Define a graph X as follows: vertices $X^0 := (G *_A H/G) \sqcup (G *_A H/H)$, edges $X^1_+ := (G *_A H)/A$ with adjacency given by $\alpha(xA) = xG$ and $\omega(xA) = xH$. Then X is a tree and the left action of $G *_A H$ on its cosets defines an action of $G *_A H$ on X, such that the quotient is a segment. Every vertex stabilizer is conjugate to either G or H, and every edge stabilizer is conjugate to A.*

This is a theorem where you really should read and understand the proof: many later results use the same ideas or refer back to it. See Exercise set 5 to familiarize with this theorem on the *torus knot groups*.

Theorem 6.6. *Let G be a group acting on a tree X such that the quotient is a segment, and let $e \in X^1$. Then $G \cong G_{\alpha(e)} *_{G_e} G_{\omega(e)}$.*

7 HNN extensions

Now for the second of two fundamental constructions in Bass–Serre theory.

Definition 7.1. Let G be a group, $A \leq G$ a subgroup and $\varphi : A \to G$ an injective homomorphism (in other words, φ is an isomorphism between two subgroups of G). The *HNN extension of G with respect to φ* is the group $G*_\varphi = \langle G, t \mid t^{-1}at = \varphi(a) : a \in A\rangle$. We call G the *base group*, A the *associated subgroup* and t the *stable letter*. We denote $B := \varphi(A)$.

As usual, generators of A suffice for the presentation.

Definition 7.2. Let T_A be a system of representatives of right cosets $A\backslash G$ and T_B of $B\backslash G$, both containing 1. A *normal form* for an element $x \in G*_\varphi$ is a decomposition $x_0 t^{\varepsilon_1} x_1 \cdots t^{\varepsilon_n} x_n$, where $x_0 \in G, x_i \in T_A \cup T_B, \varepsilon_i = \pm 1$, and t is always followed by T_B and t^{-1} is always followed by T_A, and there is no subword of the form $t^\varepsilon 1 t^{-\varepsilon}$.

See Exercise set 4 to practice normal forms on the *Baumslag–Solitar groups*.

Theorem 7.3. *Every element of $G*_\varphi$ can be written uniquely in normal form. Moreover, the existence and uniqueness of normal forms characterizes $G*_\varphi$.*

Corollary 7.4. *G embeds naturally as a subgroup of $G*_\varphi$, and t has infinite order.*

HNN extensions can be realized as subgroups of amalgamated products, which creates a relation with the previous section. They are also useful to prove a lot of embedding theorems (all due to Higman–Neumann–Neumann or a subset thereof):

Theorem 7.5. *Every countable group can be embedded in a 2-generated group that has the same type of torsion as the original group, and this construction preserves finite presentability. In particular, there exist continuum many 2-generated groups.*

Theorem 7.6. *Every countable group can be embedded in a countable group in which all elements of the same order are conjugate.*

Theorem 7.7. *Every countable group can be embedded in a countable simple, divisible group. In particular, there exist continuum many countable simple groups.*

Next, we have the second version of the Fundamental Theorem of Bass–Serre Theory, which describes HNN extensions as groups acting on trees such that the quotient is a loop (i.e., a graph with one vertex and two edges).

Theorem 7.8. *Define a graph X as follows: vertices $X^0 := G *_\varphi /G$, edges $X^1_+ := G *_\varphi /A$ with adjacency given by $\alpha(xA) = xG$ and $\omega(xA) = xtG$. Then X is a tree and the left action of $G*_\varphi$ on its cosets defines an action of $G*_\varphi$ on X, such that the quotient is a loop. Every vertex stabilizer is conjugate to G and every edge stabilizer is conjugate to A.*

See Exercise set 5 to familiarize with this theorem on the *Baumslag–Solitar groups.*

Theorem 7.9. *Let G be a group acting on a tree X such that the quotient is a loop, and let $e \in X^1$. Let $t \in G$ be such that $t \cdot \alpha(e) = \omega(e)$ (this exists). Then G_e and $t^{-1}G_e t$ are subgroups of $G_{\alpha(e)}$ isomorphic under $\varphi : g \mapsto t^{-1}gt$, and $G \cong G_{\alpha(e)}*_\varphi$.*

8 Graphs of groups and general Bass–Serre Theory

Definition 8.1. A *graph of groups* $\mathbb{G}$ consists of a graph Y, a group for each vertex and each edge of Y, where $G_{\overline{e}} = G_e$, and inclusions $\alpha_e : G_e \to G_{\alpha(e)}$. This in turns defines $\alpha_{\overline{e}} = \omega_e : G_e \to G_{\omega(e)}$.
We write $F(\mathbb{G}, Y)$ for the group

$$\langle G_v : v \in Y^0, t_e : e \in Y^1 \mid t_e t_{\overline{e}} = 1 : e \in Y^1, t_e^{-1}\alpha_e(g)t_e = \omega_e(g) : e \in Y^1, g \in G_e\rangle.$$

Given a spanning tree T of Y, the *fundamental group* of $\mathbb{G}$ (with respect to T) is the group $\pi_1(\mathbb{G}, Y, T) = \langle F(G, Y) \mid t_e = 1 : e \in T\rangle$.

To construct $\pi_1(\mathbb{G}, Y, T)$, you can start with one vertex, then inductively take the free product with the T-neighbouring vertex groups amalgamated over the corresponding edge groups, until you have covered every edge in T and thus every vertex in Y. Then you take an HNN extension for every edge of $Y^1_+ \setminus T$, with associated subgroup the edge group and isomorphism given by the two inclusions. In this sense the previous two constructions are building blocks of this more general one.

Theorem 8.2. *The group $\pi_1(\mathbb{G}, Y, T)$ is independent of the choice of T, up to isomorphism.*

In this general setting we do not have normal forms, but something still strong enough to prove theorems.

Definition 8.3. For $e \in T^1$ and $g \in G_e$, we say that $\alpha_e(g) \in G_{\alpha(e)}$ and $\omega_e(g) \in G_{\omega(e)}$ are *equivalent* (with respect to T) : notice that they represent the same element of $\pi_1(\mathbb{G}, Y, T)$. We extend this notion by transitivity so that this is an equivalence relation on the union of all vertex groups.

Definition 8.4. Any $x \in \pi_1(\mathbb{G}, Y, T)$ can be written as a product $x = x_1 \cdots x_n$, where each x_i is in some vertex group or is an edge generator. We call such a product *reduced* if no two consecutive x_i are equivalent to elements of the same vertex group, and there are no subwords of the form $t_e^{\varepsilon} t_e^{-\varepsilon}$, or $t_e^{-1} g t_e$ where g is equivalent to some element of $G_{\alpha(e)}$, or $t_e g t_e^{-1}$ where g is equivalent to some element of $G_{\omega(e)}$.

Reduced expressions exist for every element, but they are not necessarily unique. Still:

Theorem 8.5. *If $g \in \pi_1(\mathbb{G}, Y, T)$ admits a non-empty reduced expression, then $g \neq 1$.*

Corollary 8.6. *Each vertex group embeds naturally as a subgroup of $\pi_1(\mathbb{G}, Y, T)$.*

And now here is our final version of the Fundamental Theorem of Bass–Serre Theory. Both statements are more precise in the lecture notes and the proofs.

Theorem 8.7. *Let $G = \pi_1(\mathbb{G}, Y, T)$, and define a graph X as follows: vertices $X^0 := \sqcup_{v \in Y^0} G/G_v$, edges $X^1_+ := \sqcup_{e \in Y^1_+} G/G_e$, and adjacency given by $\alpha(xG_e) = xG_{\alpha(e)}$ and $\omega(xG_e) = xt_eG_{\omega(e)}$. Then X is a tree and the left action of G on its cosets defines an action of G on X, such that the quotient is isomorphic to Y. Every vertex stabilizer is conjugate to a vertex group, and every edge stabilizer is conjugate to an edge group.*

See Exercise set 6 to familiarize with this theorem on the *Generalized Baumslag–Solitar groups.*

Theorem 8.8. *Let G be a group acting on a tree X, and let Y be the quotient. Then there exists a graph of groups $\mathbb{G}$ with underlying graph Y whose fundamental group is isomorphic to G.*

Definition 8.9. The tree constructed here is called the *Bass–Serre tree* of G: the definition is really a generalization of the ones of the previous two sections.

Some related results:

Corollary 8.10. *Every finite subgroup of G is conjugate into a vertex group.*

Proposition 8.11. *A fundamental group of a finite, connected graph of groups with finite vertex groups is finitely generated and virtually free, i.e., it admits a finite-index free subgroup.*

9 Applications of Bass–Serre Theory I: Subgroups of amalgamated products

This theorem allows to understand subgroups of free products, and also of amalgamated products under additional hypotheses.

Theorem 9.1 (Kurosh)**.** *Let $(H_i)_{i \in I}$ be groups with a common subgroup A and let $H := *_A H_i$ be the corresponding amalgamated product. Let $G \leq H$ be a group intersecting trivially every conjugate of A. Then there exists a free subgroup $F \leq H$ and systems X_i of representatives of $G \backslash H / H_i$ such that*

$$G \cong F * (*_{i \in I} (*_{x \in X_i} G \cap xH_ix^{-1}))$$

10 Applications of Bass–Serre Theory II: Stallings's Theorem

From this section, only the definitions, examples, and statements are potentially relevant for the exam.

Definition 10.1. Let X be a graph. A *ray* in X is a one-way infinite path with no repeating edge. Two rays r, s are *equivalent* if for every finite subset F of X, there is a path in $X \setminus F$ connecting a vertex of r to a vertex of s. An equivalence class of rays is called an *end*.

Given a finitely generated group, we can talk about the ends of its Cayley graphs. It turns out that the number of ends is independent of the Cayley graph you choose.

Theorem 10.2 (Freudenthal)**.** *A finitely generated group has* $0, 1, 2$ *or infinitely many ends.*

The case of 0 ends encompasses precisely finite groups, the case of 1 end (e.g., $\mathbb{Z}^2$) is the mysterious one, and the other two cases (e.g., $\mathbb{Z}$ and F_2, respectively) are taken care of by Stallings's Theorem.

Definition 10.3. A group G *splits* over a subgroup A if it can be written as a free product amalgamated over A or as an HNN extension with associated subgroup A.

Theorem 10.4 (Stallings)**.** *A finitely generated group has more than one ends if and only if it splits over a finite subgroup.*

Proposition 10.5. *A finitely generated virtually free group is the fundamental group of a finite connected graph of groups with finite vertex groups.*

Corollary 10.6. *A finitely generated torsion-free virtually free group is free.*