Exercise class 1

Donnerstag, 25. Februar 2021 12:03

Topics for today:
- Group actions
- Free groups

1. Group actions

- Let G be a group, X a set.

Def: A (left-) action of G on X, $G \curvearrowright X$ is a map $G \times X \to X : g \mapsto g \cdot x$ s.t.

1) $e \cdot x = x \quad \forall x \in X$, where $e \in G$ is the identity
2) $g \cdot (h \cdot x) = (gh) \cdot x$

$\Rightarrow x \mapsto gx$ is bijective with inverse $x \mapsto g^{-1}x$

Q: Does 1) follow from 2)?

No: e.g. $g \cdot x = x_0$ for a fixed $x_0 \in X$ $\forall g, x$.

Lemma: There is a correspondence

$$\{G \curvearrowright X\} \longleftrightarrow \{\text{homomorphisms } G \to \overbrace{\mathrm{Sym}(X)}^{\text{bijections } X \to X}\}$$

Pf: $G \curvearrowright X \rightsquigarrow \varphi : G \to \mathrm{Sym}(X) : g \mapsto \{x \mapsto g \cdot x\}$

$\varphi : G \to \mathrm{Sym}(X) \rightsquigarrow g \cdot x := \varphi(g)(x)$.

Def: Let $G \curvearrowright X$.

1) $G_x := \{g \in G : g \cdot x = x\}$ the stabilizer of x.
2) $G \cdot x := \{g \cdot x : g \in G\}$ the orbit of x.

The map $G \to G \cdot x : g \mapsto g \cdot x$ is called orbit map.

The map $G \to G\cdot x : g \mapsto g\cdot x$ is called **orbit map**.

3) If $G\cdot x = X$, the action is **transitive**.

4) If $G_x = \{e\}$ $\forall\, x \in X$, the action is **free**.

5) If $G_x = G$, x is called a **fixed point** of the action.

6) The **kernel** of the action is $\bigcap_{x\in X} G_x$.

It is the kernel of φ from the lemma.

7) If the kernel is trivial, the action is **faithful**.

EX: 1) $G \curvearrowright G$: $g\cdot x = gx$. The (left) **regular** action.

It is free and transitive.

2) $G \curvearrowright G$: $g\cdot x = gxg^{-1}$.

$G_x = C_G(x) = \{g \in G : gx = xg\}$: the **centralizer** of x.

$G\cdot x = x^G$ the **conjugacy class**.

$\ker = Z(G) = \{g \in G : gx = xg\ \forall\, x \in G\}$. the **center**.

3) $H \le G$, $G \curvearrowright G/H$: $g\cdot(xH) = (gx)H$.

$G_{xH} = xHx^{-1}$, in particular $G_H = H$.

It is transitive.

Lemma: $G_{g\cdot x} = g\,G_x\,g^{-1}$.

Pf: Exercise.

– Actions of type 3) are "universal".

Def: $G \curvearrowright X$, $G \curvearrowright Y$. A map $\varphi : X \to Y$ is a **G-map** if $\varphi(g\cdot x) = g\cdot\varphi(x)$.

If it is bijective, **G-isomorphism**.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

EX: $G \to X : g \mapsto g\cdot x$ is a

EX: $G \to X : g \mapsto g\cdot x$ is a G-map, where $G \curvearrowright G$ is the regular action. $Y \xrightarrow{g} Y$

THM (Orbit-stabilizer theorem): $G \curvearrowright X$. Then the orbit map descends to a G-isomorphism

$$G/G_x \longrightarrow G\cdot x : gG_x \mapsto g\cdot x.$$

Pf: Exercise

2. Free groups

Def: G a group, $S \subset G$. $S^{-1} := \{ s^{-1} : s \in S \}$.
S generates G, denoted $G = \langle S \rangle$ if any $g \in G$ can be written as $g = s_1 \cdots s_n$, $s_i \in S \cup S^{-1}$.
$n = 0$ is allowed, $e = (\emptyset)$
We can always ensure $s_{i+1} \neq s_i^{-1}$, such expressions are called reduced.
S is a generating set for G, if $|S| < \infty$, G is finitely generated.

Def: F a group, $S \subset F$ s.t. $S \cap S^{-1} = \emptyset$ ($\Rightarrow e \notin S$).
We say that F is free with basis S if any $g \in F$ admits a $\boxed{\text{unique}}$ reduced expression $g = s_1 \cdots s_n$.
In particular $F = \langle S \rangle$

EX: 1) $\{e\}$ is free with basis $\emptyset$

2) $\mathbb{Z}$ is free with basis $\{1\}$ or $\{-1\}$.

- But are there free groups with bases of any cardinality?
Let us construct a free group with basis S, for any set S.

Def: Let A be a set called alphabet. A word on A is a finite sequence $w = a_1 \dots a_n$, $a_i \in A$, including $n=0$: ε.
The length $n = |w|$.
A subword is a subsequence of adjacent letters: $a_i a_{i+1} \dots a_{j-1} a_j$.
The set of words is denoted by A^*.

- A^* admits a semigroup structure given by concatenation, with identity ε.
There are no inverses to nonempty words, so to get a group we need to solve this.

- Let S be a set, define new symbols $\{s^{-1} : s \in S\} =: S^{-1}$ s.t. $S \cap S^{-1} = \emptyset$. Also write $(s^{-1})^{-1} = s$.
Let W be the set of words on $S \cup S^{-1}$.

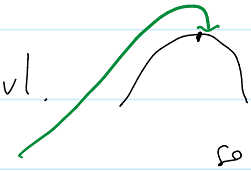
- $w, v \in W$, we say $w \sim v$ if $\exists$ a finite sequence $w = u_1, \dots, u_k = v$ s.t. u_{i+1} is obtained from u_i by inserting or deleting a subword of the form ss^{-1} ; $s \in S \cup S^{-1}$.
If no such subword ss^{-1} exists, w is called reduced.

Prop°: Any $w \in W$ is equivalent to a unique reduced word.

Pf: Existence: Remove ss^{-1} until reduced.

Uniqueness: Suppose $w \sim v$ both reduced. $w = u_1, \dots, u_k = v$.
Choose this sequence so that $\sum |u_i|$ is minimal.
Suppose by contradiction $w \neq v \Rightarrow k > 1$.

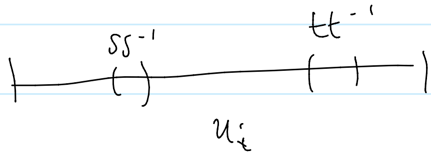
w, v reduced $\Rightarrow |w| = |u_1| < |u_2|$, $|u_{k-1}| > |u_k| = |v|$.



$\Rightarrow \exists\, 2 \le i < k$ s.t. $|u_{i-1}| < |u_i| > |u_{i+1}|$ so

$$u_{i-1} \xrightarrow[ss^{-1}]{\text{insert}} u_i \xrightarrow[tt^{-1}]{\text{delete}} u_{i+1}.$$

Case 1: are disjoint in u_i



Then you can also do

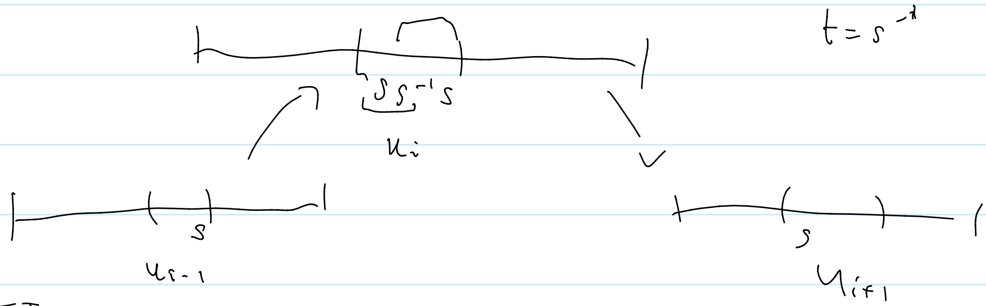
$$u_{i-1} \xrightarrow[tt^{-1}]{\text{delete}} u_i' \xrightarrow[ss^{-1}]{\text{insert}} u_{i+1}$$

But then $|u_i'| < |u_i|$ so the sequence $w = u_1, \dots, u_{i-1}, u_i', u_{i+1}, \dots, u_k$ has a smaller sum ↯.

Case 2: $(ss^{-1}) = (tt^{-1})$
Then $u_{i-1} = u_{i+1}$, so can remove u_i, u_{i+1} ↯.

Case 3: (ss^{-1}) and (tt^{-1}) have an overlap of length 1

$t = s^{-1}$



Then $u_{i-1} = u_{i+1}$ ⇝ see case 2 ↯.

THM: The set of equivalence classes of words $W/\sim$

THM: The set of equivalence classes of words $W/\sim$ with the multiplication induced by concatenation in W is a free group with basis S.

Pf. Exercise.

Not: This free group is denoted $F[S]$.

- The most important property:

THM (Universal property of free groups):

F free with basis S, G any group, $\varphi: S \to G$ a map. Then $\exists!$ homomorphism $\tilde{\varphi}: F \to G$ extending φ. Moreover this property characterizes F up to isomorphism.

Pf. We must define
$\Downarrow$
uniqueness

- $\tilde{\varphi}(s) = \varphi(s) \quad \forall\, s \in S$
- $\tilde{\varphi}(s^{-1}) = \tilde{\varphi}(s)^{-1} = \varphi(s)^{-1} \quad \forall\, s \in S$
- $\tilde{\varphi}(g = s_1 \dots s_n) = \tilde{\varphi}(s_1) \dots \tilde{\varphi}(s_n) \quad \forall\, g \in G$.

This is well defined since another expression of g just differs by

$ss^{-1} \rightsquigarrow \tilde{\varphi}(s)\tilde{\varphi}(s^{-1}) = \varphi(\cancel{s})\varphi(\cancel{s})^{-1} = e_G$

$\Rightarrow$ a homomorphism.

Let $\hat{F}$ be a group with a subset $\hat{S} \cong S$. $(\hat{s} \leftrightarrow s)$ and the above property.

$\varphi: S \to \hat{F}: s \mapsto \hat{s} \in \hat{S} \subset \hat{F} \rightsquigarrow \tilde{\varphi}: F \to \hat{F}$

$\psi: \hat{S} \to F: \hat{s} \mapsto s \in S \subset F \rightsquigarrow \tilde{\psi}: \hat{F} \to F$

$$\psi : \hat{S} \to F : \hat{s} \mapsto s \in S \subset F \rightsquigarrow \tilde{\psi} : \hat{F} \to F$$

$$\tilde{\varphi} \circ \tilde{\psi} = \widetilde{\mathrm{id}_{\hat{S}}} = \mathrm{id}_{\hat{F}} \qquad \tilde{\psi} \circ \tilde{\varphi} = \widetilde{\mathrm{id}_S} = \mathrm{id}_F .$$

$\Rightarrow$ these are isomorphisms, $F \cong \hat{F}$. $\blacksquare$

COR: Any two free groups with bases of the same cardinality are isomorphic.