Exercise class 2

Donnerstag, 11. März 2021 12:01

Correction exercise sheet 1

Exercise 1. Let X be a finite graph with n vertices. Consider the three following conditions:

(a) X is connected.] trees

(b) X has no circuit.

(c) X has $(n-1)$ positively oriented edges. i.e. counting only either e or $\bar{e}$

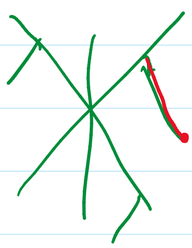
Show that any two of these conditions imply the third.

Hint. For $(a)+(b) \Rightarrow (c)$, start by showing that X has a *leaf* (i.e., a vertex of valency 1), then use induction.

Lemma: Any finite tree has a leaf.

Pf: Take a maximal path $v_1 - v_2 - \dots - v_k$ with all distinct vertices. Then v_1 and v_k are leaves.
E.g. if v_1 had a neighbour $v_0 \neq v_2$, then either we could continue the path (↯) or we have a circuit (↯). □

(a)+(b) ⇒ (c): For $n=1$ • ✓



For $n>1$, take $X' = X \setminus \{\text{leaf} + \text{edge}\}$
Then X' is still circuit-free, connected, and has $(n-1)$ vertices $\overset{\text{ind.}}{\leadsto}$ X' has $(n-2)$ p.o.edges $\leadsto$ X has $(n-1)$ p.o.edges □

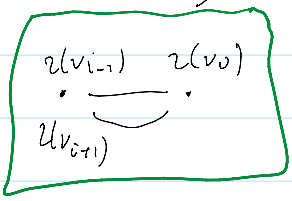
(b)+(c) ⇒ (a) Any graph decomposes as a disjoint union of connected subgraphs: the connected components.
Now write $X = \coprod_{i=1}^{k} X_i$, X_i has n_i vertices, $\sum n_i = n$. Want: $k=1$.
On X_i we can apply (a)+(b) ⇒ (c), so X_i has $(n_i - 1)$ p.o. edges.
$\leadsto$ X has $\sum_{i=1}^{k}(n_i-1) = n-k$ p.o. edges.
By (c) $k=1$. □

(c) + (a) $\Rightarrow$ (b) Any connected graph has a spanning tree, i.e., a subgraph which is a tree and touches all vertices. So let T be a spanning tree of X. By (a)&(b) $\Rightarrow$ (c) T has $(n-1)$ p.o. edges, as does X by hypothesis! So $X = T$. In particular X has no circuit. $\square$

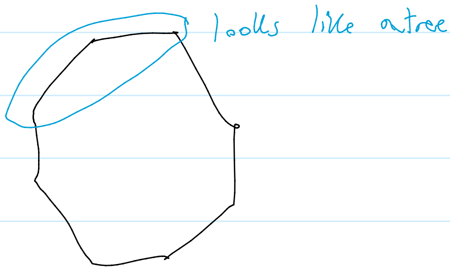
Exercise 2. Let X be a (not necessarily finite) tree, Y a connected graph. Let $\iota : Y \to X$ a morphism such that, for every $y \in Y^0$, the restriction of ι to y and its neighbours is injective: such a morphism is called *locally injective*. Show that ι is actually injective, and deduce that Y is a tree.

Does this imply that being a tree is a local property? That is, given a connected graph, can you check that it is a tree only by looking at each vertex and its neighbours?

Will show: not injective $\Rightarrow$ not locally injective.
So suppose $\exists\ x \neq y \in Y^0$ s.t. $\iota(x) = \iota(y)$. Y is connected
so take a minimal path $P: x = v_1 - v_2 - v_3 \cdots - v_{k-1} - v_k = y$.
Then $\iota(P)$ is a walk in X going from $\iota(x)$ to $\iota(y) = \iota(x)$.
Since X is a tree, $\iota(P)$ must backtrack, i.e.,
So ι is not injective around v_i.

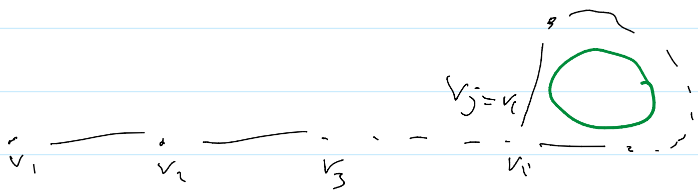


Last question: no



Note: We used that any closed walk $v_1 - v_2 \cdots - v_k = v_1$ with $v_{i-1} \neq v_{i+1}$ contains a cycle.

Pf. Let j be the first repeat vertex, and $i < j$ with $v_i = v_j$.
By minimality, the closed walk $v_i - \cdots - v_j$ is a cycle. $\square$



Exercise 3. Let X be an infinite tree.

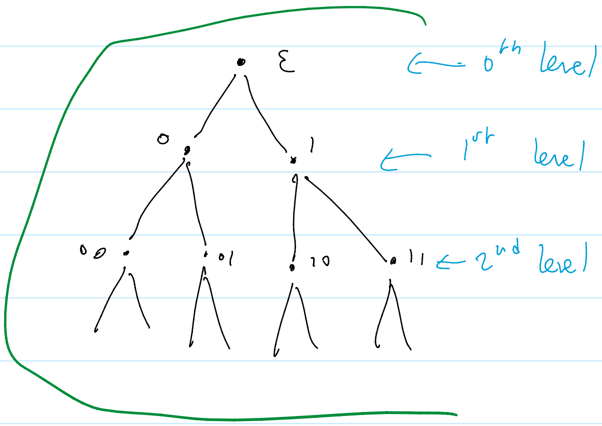
(a) Construct an example where X is such that any automorphism acts without inversion. Can you construct such an example with bounded valency?

every vertex has the same valency

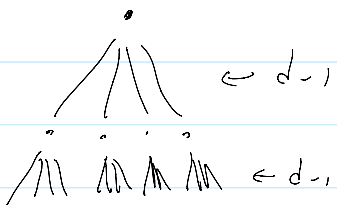
(b) Suppose that X is regular. Show that there exists some automorphism that acts with inversion.

(c) What happens if X is finite?

(a) Take X to be the binary tree.
Note ε is the only vertex of valency 2, so it is fixed by τ
$\Rightarrow$ any level is preserved by τ.
But an inversion has to invert two levels.



(b) X be d-regular. $e \in X^1$
Both $T_\pm$ are isomorphic to:



e.g. if $d = 3$, both $T_\pm$ are binary trees.
So $\exists\ \varphi : T_+ \to T_-$ isomorphism.
Extend φ to T setting $\varphi(e) = \bar{e}$.

(c) For (a) just take a finite level of the binary tree.
For (b) the only regular tree is $\bullet\!-\!\!\!-\!\bullet$.

Exercise 4. Let X be a tree and τ an automorphism. Show that τ acts without inversion on the barycentric subdivision $\mathcal{B}(X)$.

Recall: $\beta(X)^0 = X^0 \amalg X^1$, $\beta(X)^1 = X^1_- \amalg X^1_+$

$\overset{u}{\bullet}\underset{e}{\text{———}}\overset{v}{\bullet} \in X \rightsquigarrow \overset{u}{\bullet}\overset{e_-}{\text{——}}\underset{e}{\bullet}\overset{e_+}{\text{——}}\overset{v}{\bullet}$

$\tau \in Aut(X) \rightsquigarrow \tilde{\tau} \in Aut(\beta(X))$: $\tilde{\tau}|_{\beta(X)^0} = \tau$

$\tilde{\tau}(e_\pm) = \tau(e)_\pm$

Note: any edge of $\beta(X)$ has an old and a new vertex.
But $\tau \in Aut(X)$ preserves old and new vertices!
$\hookrightarrow$ it cannot invert anything.

Exercise 5. Let X be a tree, let τ, ν be automorphisms and $k \in \mathbb{Z}$ an integer.

(a) Show that $|\nu\tau\nu^{-1}| = |\tau|$.

(b) Assume that τ acts without inversion. Show that $|\tau^k| = |k| \cdot |\tau|$.

(c) Assume that τ acts with inversion. Show that $|\tau| = 1$.

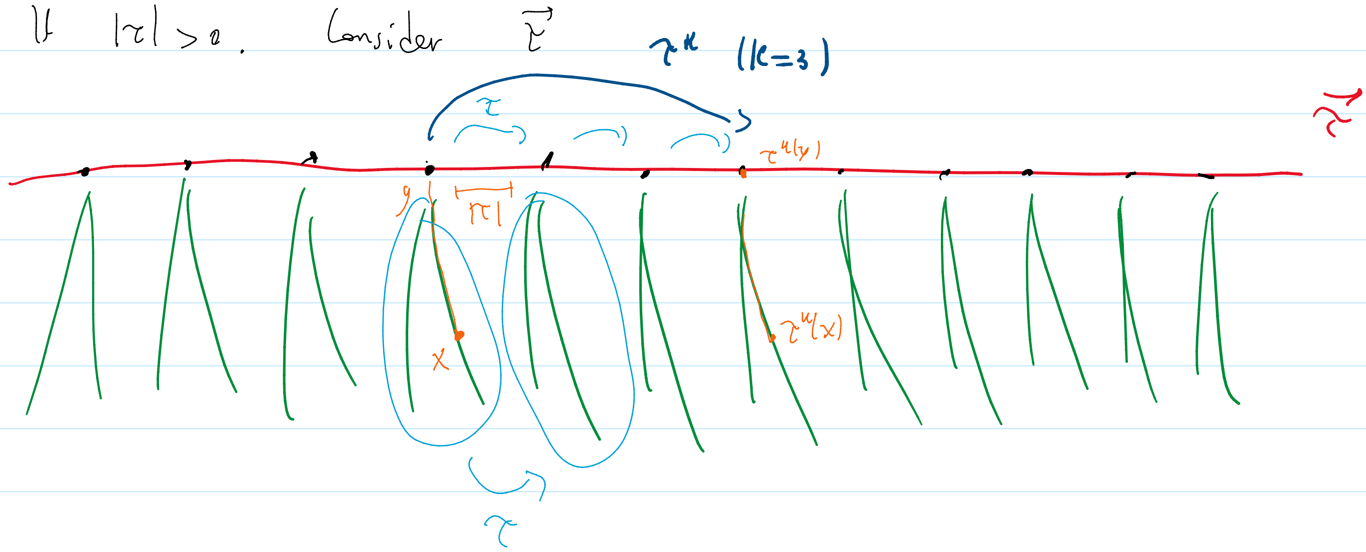
(a) $$|\nu\tau\nu^{-1}| = \min_{x \in X^0} d(x, \nu\tau\nu^{-1}(x)) =$$ d is invariant under aut.

$$= \min_{x \in X^0} d(\nu^{-1}(x), \tau\nu^{-1}(x)) =$$ change of variables $x \to \nu^{-1}(x)$

$$= \min_{x \in X^0} d(x, \tau(x)) = |\tau|.$$

(b) If $|\tau| = 0$, $\exists x : \tau(x) = x$, $\tau^k(x) = x \Rightarrow |\tau^k| = 0$. ✓

If $|\tau| > 0$. Consider $\overrightarrow{\tau}$



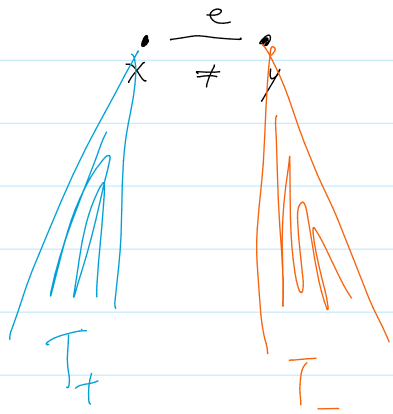
Now for $x \in \vec{\tau}$, $d(x, \tau^u(x)) = |K| \cdot |\tau|$.

For $x \notin \vec{\tau}$, let $y \in \vec{\tau}$ be the closest, then

$$d(x, \tau^u(x)) = d(x,y) + d(y, \tau^u(y)) + d(\tau^u(y), \tau^u(x)) >$$
$$> |K| \cdot |\tau|$$

So $|\tau^u| = \min_{x \in X^0} d(x, \tau^u(x)) = |K| \cdot |\tau|$

(c) Let e be s.t. $\bar{e} = \tau(e)$.
Then τ exchanges T_- with T_+
in particular τ has no fixpoints!
$\Rightarrow |\tau| > 0$.



Now $\tau(x) = y$, $d(x,y) = 1$
so $|\tau| \leq 1$
$\Rightarrow |\tau| = 1$.

Exercise 6. Draw the Cayley graph of the following pairs (G, S), where G is a group and S is a prescribed generating set. In the following items $\overline{k}$ denotes the reduction of k modulo n.

(a) $(S_3, \{(12), (123)\})$.

(b) $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \{(\overline{1}, \overline{0}), (\overline{0}, \overline{1})\})$. Think about the difference between (a) and (b).

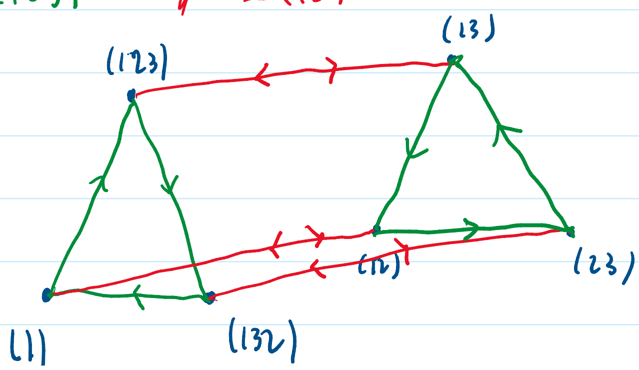
(c) $(\mathbb{Z}, S)$, where $S = \{1\}$ and where $S = \{2, 3\}$.

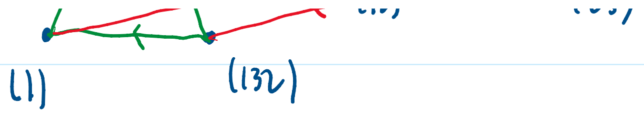
(d) $(\mathbb{Z}^2, S)$, where $S = \{(1,0), (0,1)\}$ and where $S = \{(1,0), (1,1)\}$.

(e) $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, \{(\overline{1}, \overline{0}), (\overline{0}, \overline{1})\})$.

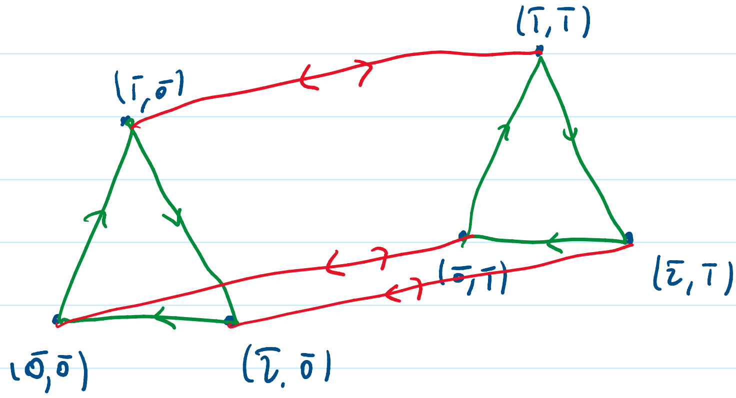
(f) $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}, \{(\overline{1}, 0), (\overline{0}, 1)\})$.

(a) / = (123) / = (12)





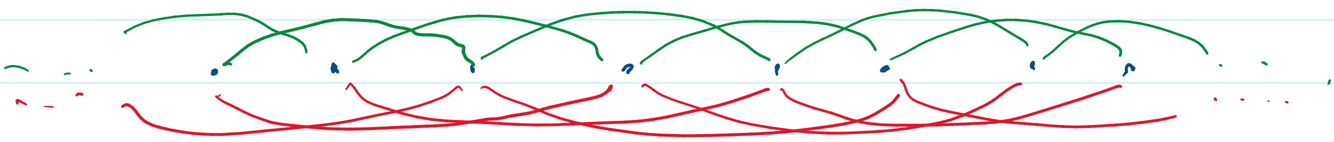
(b) / = $(\bar{1},\bar{0})$
/ = $(\bar{0},\bar{1})$



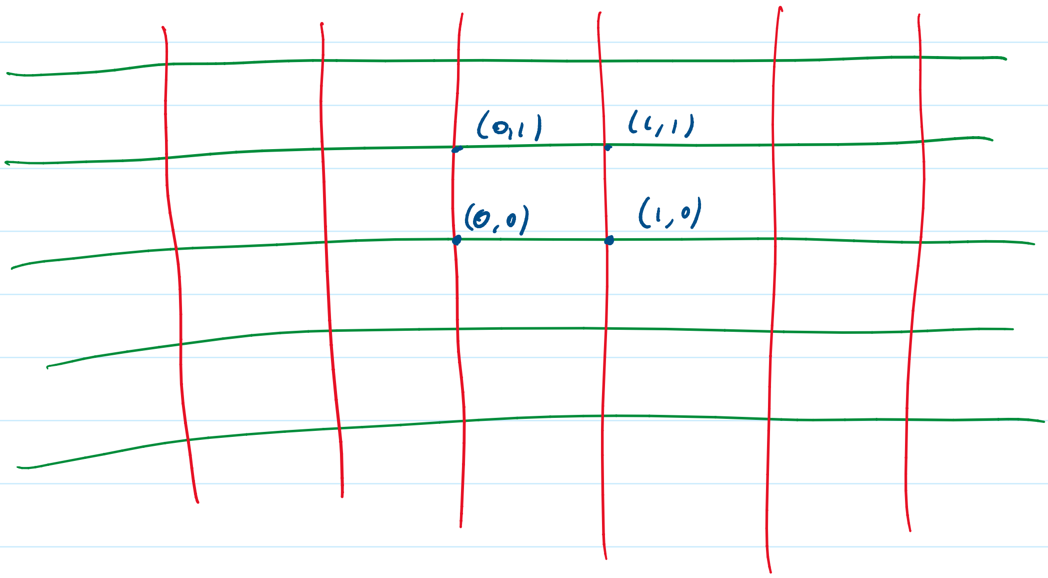
/ = 1

(c)

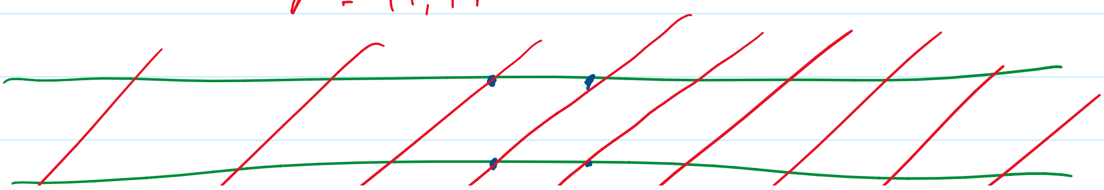
/ = 2 / = 3

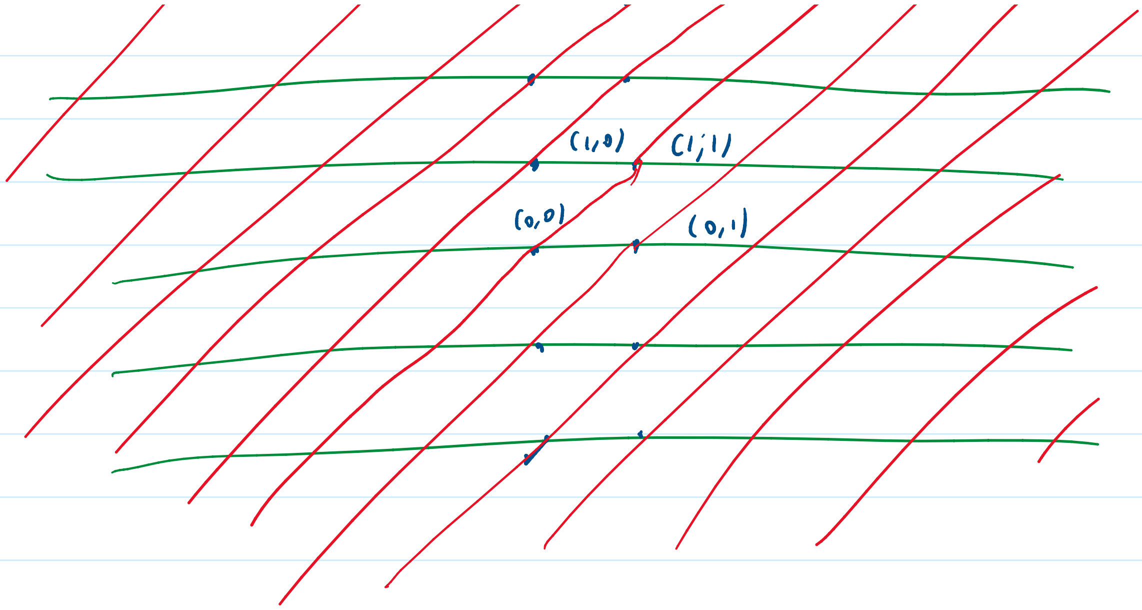


(d) / = (1,0) / = (0,1)

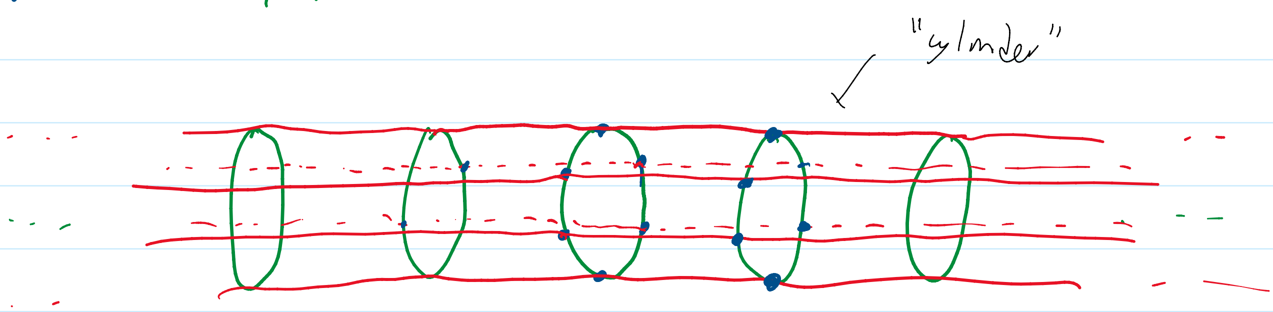


/ = (1,1)

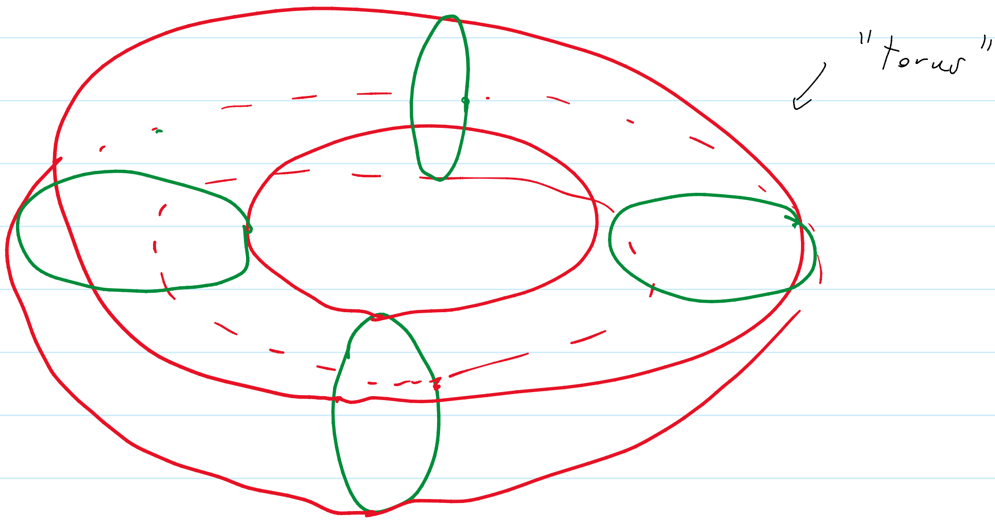




(d) = $(\bar{1}, 0)$ = $(0, 1)$



(e) = $(\bar{1}, 0)$ = $(0, \bar{1})$



Exercise 7. For each of the following properties of a pair (G, S), decide whether you can detect it by looking at the Cayley graph. What do you need to verify it? Is the underlying unoriented graph enough? Do you also need orientations? Do you also need labels?

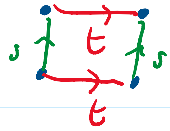
(a) S contains the identity. $\longrightarrow$ $\iff$ there are loops

(b) G contains an element of order 2. $\longrightarrow$ not really (need labels + orientations)

(c) S contains an element of order 2. $\longrightarrow$ $\iff$ double edges

(d) S contains an element of finite order. $\longrightarrow$ you need labels : $\iff$ $\exists$ a cycle with all same labels

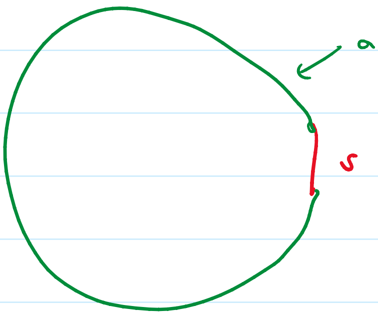
(e) S contains two elements that commute. $\to$ not really. Want to see



(f) S is not minimal: there exists a proper subset $S' \subset S$ that generates G.

$\to$ you need labels, $\iff$ $\exists$ a cycle that uses a label exactly once.

all labels in $S' = S \setminus \{s\}$



Then I can write s in terms of S'