Exercise class 3

Mittwoch, 24. März 2021 18:46

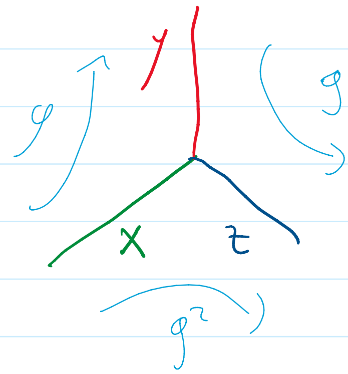
Correction exercise set 2

Exercise 1. Show that for any non-identity element $g \in F[X]$, the sequence $\{|g^n|\}_{n\geq 1}$ is strictly increasing. Deduce that non-trivial free groups are torsion-free.

$|g|>0,\quad |1|=0$

Write $g = xy = y^{-1}z$ with $|y|$ maximal

$\leadsto g^2 = xz$ reduced.



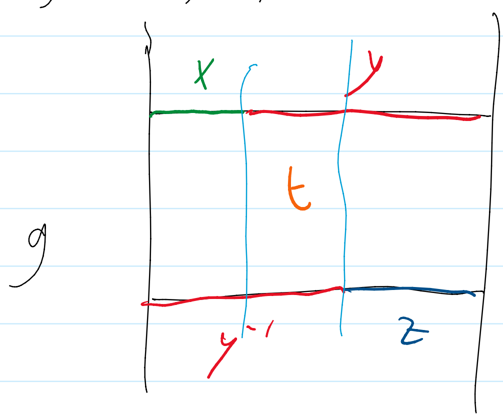
$$|g^2| = |x|+|z| = |x|+|y|+|y^{-1}|+|z|-2|y| = 2|g|-2|y|$$

So $|g^2|>|g| \iff |y|<|g|/2$

$|\cdot| = |g|/2$

Case 1: $|y| = |g|/2 \leadsto g = x\cdot y = y^{-1}\cdot z \Rightarrow x = y^{-1} \Rightarrow g = y^{-1}\cdot y$ ↯.

Case 2: $|y| > |g|/2$



$|t|>0$

$y = t\cdots \qquad y^{-1} = \cdots t$

$\Downarrow$

$y = t^{-1}\cdots$

$t = t^{-1}$

↳ impossible! $\quad t = x_1\cdots x_n$, $t^{-1} = x_n^{-1}\cdots x_1^{-1}$ $\quad x_n = x_1^{-1},\ x_{n-1} = x_2^{-1},\dots$

$\dots$ $\begin{cases} n \text{ even: } & x_{\frac{n}{2}} = x_{\frac{n}{2}+1}^{-1} \quad ↯ \text{ (not reduced)} \\ n \text{ odd: } & x_{\frac{n+1}{2}} = x_{\frac{n+1}{2}}^{-1} \quad ↯ \ (x\neq x^{-1}\ \forall x\in X). \end{cases}$

$\leadsto$ We conclude that $|g^2|>|g|$.

Same argument: $|g^{n+1}|>|g^n|$.

Exercise 2. Let $x \in X$. Show that the centralizer in $F[X]$ of x is the group generated by x. Deduce that free groups of rank at least 2 have trivial center.

Let $\omega \in C_{F[X]}(x)$. If $|\omega| = 1$, $\omega = x^{\pm 1}$.

Suppose by induction: if $|\omega| < n$, $\omega = x^k$ $k \in \mathbb{Z}$.

Let $|\omega| = n$. $\quad \underset{|\cdot| = n}{\omega} = x \underset{\text{reduced}}{\omega} x^{-1} \rightsquigarrow$ some reduction on RHS

Wlog reduction at the end $\rightsquigarrow$ $\omega = \omega' \cdot x$

$$\omega' x = \omega = x \omega x^{-1} = x \omega' \Rightarrow \text{by induction } \omega' = x^k$$
$$\Rightarrow \omega = x^{k+1}.$$

$$Z(F[X]) \leq C_{F[X]}(x) \cap C_{F[X]}(y) = \langle x \rangle \cap \langle y \rangle = \{1\}.$$
($y \neq x$)

Exercise 3. Show that the abelianization of $F[X]$ is isomorphic to $\mathbb{Z}[X]$ (= $\bigoplus_X \mathbb{Z}$): the free abelian group with basis X. Use this to give another proof that any two bases of $F[X]$ have the same cardinality.

Hint. Recall from Algebra 1 that free abelian groups are also characterized by a universal property.

THM (Universal property of free abelian groups):

F free abelian with basis X, G any abelian group, $\varphi : X \to G$ a map. Then $\exists!$ homomorphism $\tilde{\varphi} : F \to G$ extending φ. Moreover this property characterizes F up to isomorphism.

Will prove: $A = Ab(F[X])$ has this universal property, where

$$X \longrightarrow A : x \longmapsto x[F,F] \in A$$

So let G be abelian, $\varphi : X \to G$ a map.

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Universal p. of $F \Rightarrow \exists! \ \tilde{\varphi} : F[X] \to G$ hom.

Universal p. of $Ab \Rightarrow \tilde{\varphi}$ factors through $\hat{\varphi} : A \to G$.

$$\begin{tikzcd} & F[X] \arrow[dr] \arrow[d, "\exists! \tilde{\varphi}"'] & \\ X \arrow[ur] \arrow[r, "\varphi"'] & G & A \arrow[l, "\exists! \hat{\varphi}"] \end{tikzcd}$$

By characterization

$A \cong \mathbb{Z}[X]$.

Rmk: Usually it is hard to determine whether $g \in [G,G]$

For the free group, really easy!

$w = x_1 x_2 x_3 \cdots x_n \longrightarrow A$

groups together all occurrences of each $x \in X$

So $w \in [F,F] = \operatorname{Ker} \iff \forall x \in X$, the exponents of x in w sum to 0.

EX: $w = a^{-1} b^{7} a^{3} b^{-5} a^{-2} b^{-2}$

Exercise 4. Let G be a group, F be a free group and $\varphi : G \to F$ a surjective homomorphism. Show that there exists a homomorphic section, that is, a homomorphism $\sigma : F \to G$ such that $\varphi\sigma = id_F$. Deduce that G contains a subgroup isomorphic to F.

Let $F = F[X]$, choose $\sigma(x) \in \varphi^{-1}(x) \neq \emptyset$ (surj.)

This extends to $\sigma : F \to G$ (univ. p.)

This extends to $\sigma : 1 \longrightarrow U$ (univ. p.)

$$\varphi\sigma(x) = x \quad \forall x \in X \Rightarrow \varphi\sigma = id_F.$$

Exercise 5. Let $X = \{a, b\}$, and let $X_k = \{b, aba^{-1}, a^2ba^{-2}, \ldots, a^{k-1}ba^{1-k}, a^k\}$ for $k > 1$.

(a) Show that X_k generates the kernel of the homomorphism $F[X] \xrightarrow{\varphi_k} \mathbb{Z}/k\mathbb{Z}$ defined on the basis by $a \mapsto 1, b \mapsto 0$.

(b) Show that the subgroup generated by X_k is free with basis X_k.

Deduce that a free group of finite rank contains free subgroups of finite index and arbitrarily large finite rank, and these subgroups can be chosen to be normal. Can you find a free subgroup of infinite rank?

(a) $X_k \subset \operatorname{Ker}\varphi_k \Rightarrow \langle X_k \rangle \leq \operatorname{Ker}\varphi_k.$

$\omega \in \operatorname{Ker}\varphi_k$. $\quad \omega = a^{\varepsilon_1} b^{\delta_1} a^{\varepsilon_2} b^{\delta_2} \cdots a^{\varepsilon_n} b^{\delta_n}$

$\delta_1, \varepsilon_2, \ldots, \delta_{n-1}, \varepsilon_n \neq 0$

$$\omega = a^{\varepsilon_1} b^{\delta_1} (a^{-\varepsilon_1} a^{\varepsilon_1}) a^{\varepsilon_2} b^{\delta_2} (a^{-\varepsilon_1-\varepsilon_2} a^{\varepsilon_1+\varepsilon_2}) \cdots$$

$$\cdots b^{\delta_n} a^{-\varepsilon_1-\cdots-\varepsilon_n} a^{\varepsilon_1+\cdots+\varepsilon_n}$$

Now $\varepsilon_1 + \cdots + \varepsilon_n = \varphi_k(\omega) \equiv 0 \mod k \Rightarrow a^{\varepsilon_1+\cdots+\varepsilon_n} \in \langle a^k \rangle \leq \langle X_k \rangle$

For the other terms: $\quad a^{\varepsilon} b^{\delta} a^{-\varepsilon} \qquad \varepsilon = m.k + r$

$$a^{mk} \cdot \underbrace{(a^r b a^{-r})^{\delta}}_{\in \langle X_k \rangle} a^{-mk} \qquad 0 \leq r < k$$

⚠ It is true that $aba^{-1}b^{-1} \in \langle X_k \rangle$

$\rightsquigarrow$ get a map $\mathbb{Z}^2 \longrightarrow \mathbb{Z}/k\mathbb{Z}$

But $\operatorname{Ker} F[a,b] \longrightarrow \mathbb{Z}^2$ is infinitely generated!

But $\ker F(a,b) \to \mathbb{Z}^2$ is infinitely generated!
even if it is normally finitely generated
(see ex sheet 3)

(b) $X_i = a^i b a^{-i}$ $0 \le i < k$, $X_k = a^k$.

$w \in \langle X_k \rangle$ can be written $w = X_{i_1}^{n_1} \cdots X_{i_\ell}^{n_\ell}$ $i_j \neq i_{j+1}$
$n_j \in \mathbb{Z} \setminus \{0\}$

WTS: if $\ell \geq 1$, $w \neq 1$.

Will show: no occurrence of b gets reduced in the X-reduced form of w.

$W = X_{i_1}^{n_1} \cdots X_{i_\ell}^{n_\ell}$ is reduced in X_k, not in X!

Globally possible $X_{i_1}^{n_1} = \cdots a^{-1}$, $X_{i_2}^{n_2} = a \cdots$

If we show this, then either $w = a^{k \cdots} \neq 1$
or there is ~~some~~ b in the reduced expression of $w \neq 1$

Will prove this by induction on ℓ.

$\underline{\ell = 1}$ ✓ $X_{i_1}^{n_1} = \begin{cases} a_k^{n_1} & \text{if } i_1 = k \\ a^{i_1} b^{n_1} a^{-i_1} \end{cases}$ ✓

$\underline{\ell > 1}$: By induction, no b gets reduced in $X_{i_2}^{n_2} \cdots X_{i_\ell}^{n_\ell}$

So only possible problem at $X_{i_1}^{n_1} \cdot X_{i_2}^{n_2}$

If $X_{i_1} = a^k$, nothing to do! So $X_{i_1}^{n_1} = a^{\varepsilon} b^{\delta} a^{-\varepsilon}$

If this occurrence is cancelled then $X_{i_2}^{n_2} \cdots X_{i_\ell}^{n_\ell} \overset{X\text{-red.}}{=} a^{\varepsilon} b^{\pm 1} \cdots$

If this occurence is cancelled then $x_{i_2}^{n_2} - x_{i_\ell}^{n_\ell} \overset{X\text{-red.}}{=} a^{\varepsilon} b^{\pm 1} \text{———}$

But by induction, no occurence of b is cancelled in

$\Rightarrow \quad x_{i_2} = a^{-\varepsilon} b a^{-\varepsilon}$ But this means $i_1 = i_2$

Contradicts the hypotheses.

Exercise 6. Let G be a finite group of order n. Explain how to construct a finite presentation of G. How many generators does it have in terms of n? How many relators? What is the length of the relators?

Apply this to the group $\mathbb{Z}/n\mathbb{Z}$, and compare the presentation you obtain to the standard presentation $\langle x \mid x^n \rangle$.

$S = \{ s_g : g \in G \} \xrightarrow{\#} n$

$R = \{ s_g \cdot s_h = s_{gh} \mid g, h \in G \} \xrightarrow{\#} n^2, \quad \text{length} = 3$

$G \cong \langle S \mid R \rangle : \quad F_S / \langle\langle R \rangle\rangle \xrightarrow{\cong} G$

$s_g \longmapsto g$

EX: $\mathbb{Z}/4\mathbb{Z} : \quad \langle s_0, s_1, s_2, s_3 \mid s_0 s_1 = s_1 \quad s_1 s_3 = s_0 \quad s_1 s_2 = s_3 \quad \ldots \rangle$

Sim. for $\mathbb{Z}/_n\mathbb{Z} = \langle x \mid x^n = 1 \rangle$

Exercise 7. Let $G = \langle S_G \mid R_G \rangle$ and $H = \langle S_H \mid R_H \rangle$ be two groups defined by finite presentations. For each of the constructions below, explain how to construct a finite presentation using the given ones of G and H.

(a) The direct product $G \times H$. Apply this to the group $\mathbb{Z}^2$.

(b) More generally, a semidirect product $G \rtimes_\varphi H$, where $\varphi : H \to Aut(G)$ is a homomorphism. Apply this to the dihedral group D_n.

Remark. Recall that $G \rtimes_\varphi H$ is the group with underlying set the Cartesian product $G \times H$, and with composition defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 \varphi(h_1)(g_2), h_1 h_2).$$

$$(g_1, h_1)(g_2, h_2) = (g_1\varphi(h_1)(g_2), h_1h_2).$$

(a) $S = S_G \amalg S_H$

$R = R_G \amalg R_H \amalg \{ [g,h]=1 : g \in S_G, h \in S_H \}$

$\mathbb{Z}^2 = \langle x, y \mid [x,y]=1 \rangle$

(b) $S = S_G \amalg S_H$

$R = R_G \amalg R_H \amalg \{ \quad : g_1, g_2 \in S_G, \ h_1, h_2 \in S_H$ writing $\varphi(h_1)(g_2)$ in terms of $S_G \}$

also enough

$\{ hgh^{-1} = \underbrace{\varphi(h)(g)}_{\text{written in } S_G} : g \in S_G, h \in S_H \}$

$D_n = \langle a, t \mid a^n = 1, t^2 = 1, tat^{-1} = a^{-1} \rangle$

$= \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \qquad \varphi(t)(a) = a^{-1}.$