Exercise class 4

Mittwoch, 14. April 2021 16:41

# Correction exercise set 3

The following exercise is a typical application of the ping-pong lemma. The end goal is to show that $\mathrm{SL}_2(\mathbb{Z})$ is *virtually free*, that is, it contains a free subgroup of finite index.

**Exercise 1.** Let $G \leq \mathrm{SL}_2(\mathbb{Z})$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by

$$x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}; \; y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

(a) Use the ping-pong lemma on the action of $G$ on $\mathbb{Z}^2$ to show that $G$ is free with basis $\{x, y\}$.

(b) Show that $G$ contains the modulo 4 congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, namely $\mathrm{SL}_2(\mathbb{Z})_4 := \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv I \mod 4\}$.

(c) Deduce that $\mathrm{SL}_2(\mathbb{Z})$ is virtually free. Is $\mathrm{SL}_2(\mathbb{Z})$ free?

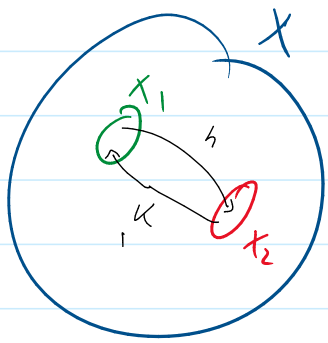
*Hint.* For (a), think about what happens to the absolute values of the entries of a point in $\mathbb{Z}^2$ after applying $x$ or $y$.

(a) Ping pong: $G \curvearrowright X$, $H, K \leq G$ generate $G$. $\underbrace{X_1, X_2}_{\text{disjoint}} \subset X$ s.t.

$\forall\ h \in H \setminus \{1\}$ $h(X_1) \subset X_2$

$\forall\ k \in K \setminus \{1\}$ $k(X_2) \subset X_1$

Then $G \cong H * K$.




So we want to find $X_1, X_2 \subset \mathbb{Z}^2$, $X_1 \cap X_2 = \emptyset$

s.t. $x^k(X_1) \subset X_2$ $\forall\ k \in \mathbb{Z} \setminus \{0\}$

$y^k(X_2) \subset X_1$ $\forall\ k \in \mathbb{Z} \setminus \{0\}$.

$$x^k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 2kb \\ b \end{pmatrix} \qquad y^k \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b + 2ka \end{pmatrix}$$

If $|a| > |b|$, $k \neq 0$, then $|b + 2ka| \geq 2|k|\,|a| - |b| \geq 2|a| - |b| > |a|$.

If $|a| < |b|$, $k \neq 0$, then $|a + 2kb| \geq \cdots > |b|$.

Thus setting $X_1 = \{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2 : |a| < |b| \}$ we have

Thus setting $X_1 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2 : |a| < |b| \right\}$ we have
$X_2 = \left\{ \quad '' \quad |a| > |b| \right\}$ $X_1 \cap X_2 = \phi$.

$x^k(X_1) \subset X_2, \qquad y^k(X_2) \subset X_1, \qquad \forall\, k \neq 0.$

By ping-pong $G = \langle x, y \rangle \cong \langle x \rangle * \langle y \rangle \cong F_2$.

(c) $[\underset{\uparrow \text{ free}}{SL_2(\mathbb{Z}) : G}] \leqslant [SL_2(\mathbb{Z}) : SL_2(\mathbb{Z})_4] = |SL_2(\mathbb{Z}) / SL_2(\mathbb{Z})_4| =$

$= |SL_2(\mathbb{Z}/4\mathbb{Z})| < \infty$. finite-index

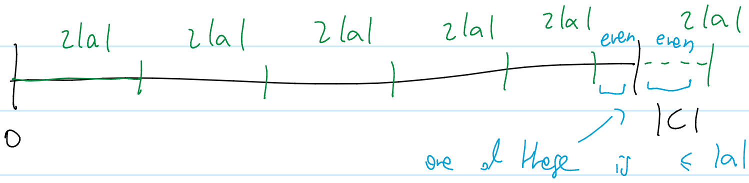
It is $\underline{\text{not}}$ free because it has torsion (e.g $-I$) (ex 1 ex set 2)

(b) We will show $G = \overbrace{\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a, d \equiv 1 \bmod 4 \\ b, c \equiv 0 \bmod 2 \end{array} \right\}}^{H}$

$\qquad\qquad\qquad\qquad \begin{array}{cc} \cap\!| & \vee\!| \\ SL_2(\mathbb{Z}) & SL_2(\mathbb{Z})_4 \end{array}$

Check: $H$ is a group.
$x, y \in H \implies G \in H$.

To show: $H \leqslant G$. Idea: start with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$, and multiply on the left by $x^k, y^k$ until you get $I$.
Will focus on $\begin{pmatrix} a \\ c \end{pmatrix}$. $a$ is odd, $c$ is even. $\implies |a| \neq 0$, $\neq |c|$




$\underline{|a| < |c|}$ :

This shows: $\exists\ k \in \mathbb{Z}$ s.t. actually, $< |a|$ since $|c|$ even
$|c + 2ka| \leq |a|$.
So $y^k \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a \\ c + 2ka \end{pmatrix}$ falls in case 2.

$\underline{|c| < |a|}$ : Either $|c| = 0$, then stop. Or: $\exists\ k$ s.t.
$|a + 2kc| < |c|$. So $x^k \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} a + 2kc \\ c \end{pmatrix}$ falls in case 1.

At some point this stops $\rightsquigarrow |c|=0$. So we found $g \in G$ s.t. $\underbrace{g}_{G \leq H} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in H} = \underbrace{\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}}_{\in H \leq SL_2(\mathbb{Z})}$. $\Rightarrow a' \cdot d' = 1$

$\Rightarrow a' = d' = \pm 1 \overset{\in H}{\Rightarrow} a' = d' = 1 \Rightarrow \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \in \langle X \rangle$.

So $g \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

**Exercise 2.** Let $G$ and $H$ be groups. Consider the natural homomorphism $\varphi : G * H \to G \times H$, that is, the unique homomorphism such that $\varphi|_G : G \to G \times H : g \to (g, 1)$ and $\varphi|_H : H \to G \times H : h \mapsto (1, h)$.

(a) Show that the kernel of $\varphi$ is free with basis $\overbrace{\{[g, h] : g \in G \setminus \{1\}, h \in H \setminus \{1\}\}}^{X}$.

(b) Deduce that the commutator subgroup of $F_2$ is free of infinite rank (compare with the last question of Exercise 5 in Exercise set 2).

(c) Deduce that a free product of two finite groups is virtually free. Can it be free?



(d) Use (c) to give another proof that $SL_2(\mathbb{Z})$ is virtually free.

(a) $K := \langle X \rangle$. $X \subset \operatorname{Ker} \varphi \Rightarrow K \leq \operatorname{Ker} \varphi$.

Moreover, $K$ is normal!

- $[g,h]^{-1} = [h,g]$
- $x \in G$, $x[g,h]x^{-1} = [xg,h] \cdot [x,h]^{-1} \in K$
- $y \in H$, $y[g,h]y^{-1} = [g,y]^{-1} \cdot [g,yh] \in K$

$\Rightarrow K$ is normalized by both $G$ and $H \Rightarrow K$ is normal.

So $G * H / K$ is a group in which $G$ and $H$ commute, so $\operatorname{Ker} \varphi \subset K$.

Now show: $\ker\varphi$ is freely generated by $X$, so take

$w = [g_1, h_1]^{\varepsilon_1} \cdots [g_n, h_n]^{\varepsilon_n}$ reduced:
- $g_i \in G \setminus \{1\}$, $h_i \in H \setminus \{1\}$, $\varepsilon_i \pm 1$
- $[g_{i+1}, h_{i+1}]^{\varepsilon_{i+1}} \neq [g_i, h_i]^{-\varepsilon_i}$

To show: (if $n > 0$) $w \neq 1$. ] That is, the identity admits a unique reduced expression (*)

**Claim:** When reducing $w$ in $G * H$, the last two letters survive.
That is, the normal form of $w$ in $G * H$ ends with
$g_n^{-1} h_n^{-1}$ (if $\varepsilon_n = 1$) or $h_n^{-1} g_n^{-1}$ (if $\varepsilon_n = -1$)
In particular, $w \neq 1$.

**pf:** If $n = 1$, done. Say $n > 1$, and true up to $(n-1)$.
Assume wlog $\varepsilon_{n-1} = 1$ (otherwise same proof)
By induction $w' = [g_1, h_1]^{\varepsilon_1} \cdots [g_{n-1}, h_{n-1}]^{\varepsilon_{n-1} = 1}$ ends with $g_{n-1}^{-1} h_{n-1}^{-1}$.

**Case 1:** $\varepsilon_n = 1$.

$$w = w'[g_n, h_n] = \cdots \underset{G}{g_{n-1}^{-1}} \underset{H}{h_{n-1}^{-1}} \underset{G}{g_n} \underset{H}{h_n} \underset{G}{g_n^{-1}} \underset{H}{h_n^{-1}}$$ is in normal form. ✓

**Case 2:** $\varepsilon_n = -1$

$$w = w'[g_n, h_n]^{-1} = \cdots g_{n-1}^{-1} h_{n-1}^{-1} h_n g_n h_n^{-1} g_n^{-1}$$ ✓

If $h_{n-1} \neq h_n$, then $w = \cdots \underset{G}{g_{n-1}^{-1}} \underset{H \neq 1}{(h_{n-1}^{-1} h_n)} \underset{G}{g_n} \underset{H}{h_n^{-1}} \underset{G}{g_n^{-1}}$

is in normal form

Else $h_{n-1} = h_n$, then $w = \cdots \underset{G}{(g_{n-1}^{-1} g_n)} \underset{H}{h_n^{-1}} \underset{G}{g_n^{-1}}$ ✓

Now if $g_n = g_{n-1}$, then $[g_n, h_n]^{\varepsilon_n} = [g_{n-1}, h_{n-1}]^{-\varepsilon_{n-1}}$ ↯ (*) above

So $g_n \neq g_{n-1} \Rightarrow$ the above is in normal form $\blacksquare$

(b) Consider $\varphi : F_2 = \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. By exercise 3 in exercise set 2, this is the abelianization map, so
$\ker\varphi = [F_2, F_2]$ free w/ basis $\{\underbrace{[a^k, b^\ell]}_{\text{infinite}} : k, \ell \in \mathbb{Z} \setminus \{0\}\}$.
(a)

(c) $G, H$ finite. $G * H$ contains $\ker\varphi$ as a free subgroup of index
$[G * H : \ker\varphi] = |\operatorname{im}\varphi| = |G \times H| = |G| \cdot |H| < \infty$

It cannot be free because it has torsion ($G, H$ are finite)

(d) By (c), $PSL_2(\mathbb{Z}) \overset{\text{class}}{\cong} \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ is v free.

$$\begin{array}{ccc} SL_2(\mathbb{Z}) & \xrightarrow{\pi} & PSL_2(\mathbb{Z}) \\ \text{\rotatebox{90}{$\leq$} f.i.} & & \text{\rotatebox{90}{$\leq$} f.i.} \quad \exists \text{ by (c)} \\ \pi^{-1}(G) & \longrightarrow & G \end{array}$$

$\pi^{-1}(G)$ cannot be free because it contains $\{\pm I\} = \ker \pi$.

By exercise 4 in ex. set 2, $\exists$ a section $\sigma : G \to \pi^{-1}(G)$.

Then $\sigma(G) \overset{\text{f.i.}}{\leq} \pi^{-1}(G)$ is free (f.i. $= 2 = |\ker\pi|$).

$\left[ \pi^{-1}(G) = \ker\pi \times \sigma(G) \right] \Rightarrow \sigma(G) \overset{\text{f.i.}}{\leq} SL_2(\mathbb{Z})$.

**Exercise 3.** Consider the semidirect product $G := F_2 \rtimes_{\varphi} \mathbb{Z}/2\mathbb{Z} = \overbrace{\langle a, b, t \mid tat^{-1} = b, t^2 = 1 \rangle}^{\text{see. ex. class 3}}$. Show that $G \cong \underbrace{\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}}_{}$.

$\varphi : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut}(F_2) : 1 \mapsto \left\{ \begin{array}{l} a \mapsto b \\ b \mapsto a \end{array} \right\}$

$\langle \alpha, \tau \mid \tau^2 = 1 \rangle$

$$\begin{array}{ccc} F_2 \rtimes \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \\ a & \longmapsto & \alpha \\ b & \longmapsto & \tau\alpha\tau^{-1} \\ t & \longmapsto & \tau \end{array}$$

$$\begin{array}{ccc} F_2 \rtimes \mathbb{Z}/2\mathbb{Z} & \longleftarrow & \mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \\ a & \longmapsfrom & \alpha \\ t & \longmapsfrom & \tau \end{array}$$

$a \longleftarrow\!\!\!\mid \alpha$

$t \longleftarrow\!\!\!\mid \tau$

**Exercise 4.** Let $A, G, H$ be groups, and $\iota_G : A \to G, \iota_H : A \to H$ injective homomorphisms. Let $G *_A H$ be the corresponding amalgamated product, with the canonical injective homomorphisms $j_G : G \to G *_A H$ and $j_H : H \to G *_A H$.

$= \langle G, H \mid \iota_G(a) = \iota_H(a) : a \in A \rangle$

(a) Show that $G *_A H$ enjoys the following universal property. For every group $K$ and every pair of homomorphisms $\varphi_G : G \to K, \varphi_H : H \to K$ such that $\varphi_G \circ \iota_G = \varphi_H \circ \iota_H : A \to K$, there exists a unique homomorphism $\varphi : G *_A H \to K$ such that $\varphi_G = \varphi \circ j_G$ and $\varphi_H = \varphi \circ j_H$.

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_G"] \arrow[d, hook, "\iota_H"'] & G \arrow[d, hook, "j_G"] \arrow[ddr, bend left, "\forall \varphi_G"] & \\
H \arrow[r, hook, "j_H"] \arrow[drr, bend right, "\forall \varphi_H"'] & G *_A H \arrow[dr, dashed, "\exists! \varphi"] & \\
& & K
\end{tikzcd}$$

(b) Show that this universal property characterizes $G *_A H$. That is, show that if $L$ is a group with homomorphisms $\hat{j}_G : G \to L, \hat{j}_H : H \to L$ such that $\hat{j}_G \circ \iota_G = \hat{j}_H \circ \iota_H : A \to L$, and $L$ has the universal property above, then there exists a canonical isomorphism $L \cong G *_A H$.

(a) Uniqueness: $\varphi_G = \varphi \circ j_G$ defines uniquely $\varphi$ on $j_G(G)$, $j_H(H)$ generate $G *_A H$

$\varphi_H = \varphi \circ j_H$

Existence: Define $\tilde{\varphi} : G * H \to K$ : $\tilde{\varphi}|_G = \varphi_G$, $\tilde{\varphi}|_H = \varphi_H$

Homomorphism ✓

Moreover $\tilde{\varphi}(\iota_G(a)) = \varphi_G(\iota_G(a)) = \varphi_H(\iota_H(a)) = \tilde{\varphi}(\iota_H(a))$.

So $\tilde{\varphi}$ descends to $\varphi : G *_A H \to K$.

(b)

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_G"] \arrow[d, "\iota_H"'] & G \arrow[d, "j_G"] \arrow[ddr, bend left, "\hat{j}_G"] & \\
H \arrow[r, hook, "j_H"'] \arrow[drr, bend right, "\hat{j}_H"'] & G *_A H \arrow[dr, dashed, "\exists \psi"] & \\
& & L
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, hook, "i_G"] \arrow[d, "\iota_H"'] & G \arrow[d, "\hat{j}_G"] \arrow[ddr, bend left, "j_G"] & \\
H \arrow[r, "\hat{j}_H"'] \arrow[drr, bend right, "j_H"'] & L \arrow[dr, dashed, "\exists \psi"] & \\
& & G *_A H
\end{tikzcd}$$

by (a) ⟶

by hyp.

Composing $\psi$ and $\psi_1$ by uniqueness we get the identity.

Composing $\varphi$ and $\psi$, by uniqueness we get the identity.

**Exercise 5.** Let $m, n \in \mathbb{Z}$, and consider the *torus knot group* $K_{m,n} := \langle a, b \mid a^m = b^n \rangle$.

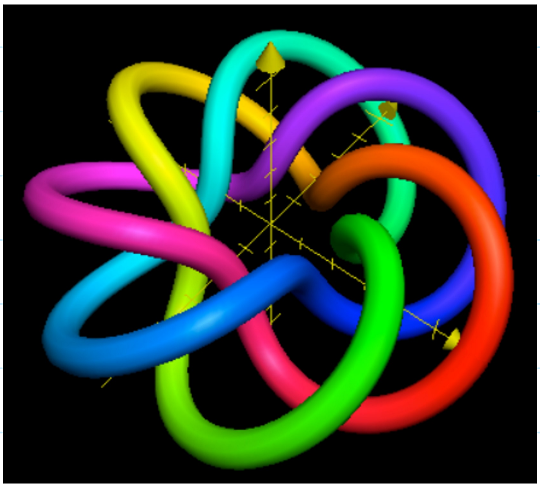
(a) Show that $K_{m,n} \cong K_{-m,n} \cong K_{m,-n} \cong K_{n,m}$.

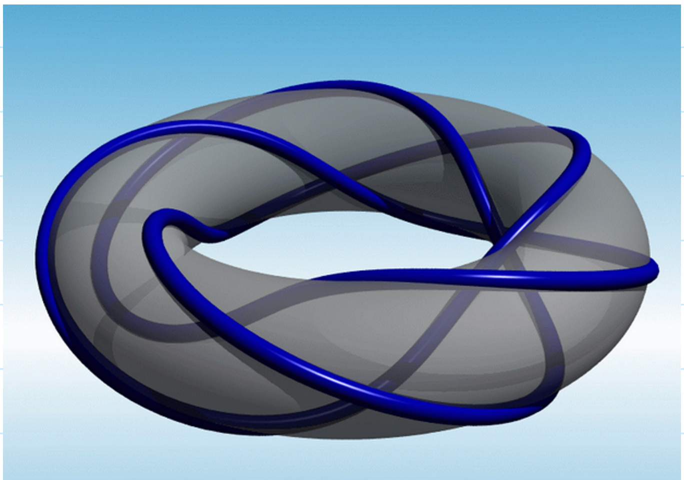
(b) Express $K_{m,n}$ as an amalgamated free product, find transversals, and describe the normal forms with respect to these.

(c) Prove that the amalgamated subgroup $\langle a^m \rangle = \langle b^n \rangle$ is contained in the center of $K_{m,n}$.

(d) Let $m = 7$ and $n = -6$. Find the normal forms of the following elements:

- $a^{-3}b^2(ab)^3b^{-5}$.
- $b^{11}a^{23}b^{-1}ab^{-11}$.
- $(ab)^{100}$.





(a)
$$K_{m,n} = \langle a, b \mid a^m = b^n \rangle$$
$$K_{-m,n} = \langle x, y \mid x^{-m} = y^n \rangle$$
$$K_{n,m} = \langle \beta, \alpha \mid \beta^n = \alpha^m \rangle$$

$a \mapsto x^{-1}$, $b \mapsto y$

$a \mapsto \alpha$, $b \mapsto \beta$

(b)
$$K_{m,n} = G *_A H = \underset{\langle a \rangle}{\overset{\shortparallel}{\mathbb{Z}}} *_{\mathbb{Z}} \underset{\langle b \rangle}{\overset{\shortparallel}{\mathbb{Z}}}$$

$\langle a^m \rangle \leftrightarrow \langle b^n \rangle$

$$\tau_A = \{e, a, a^2, \ldots, a^{|m|-1}\} \qquad \tau_B = \{e, b, \ldots, b^{|n|-1}\}.$$

some set $(x_i)_{i \in I}$ s.t. $G = \coprod x_i A$, containing identity

A-normal form: $x_0 \underset{\tau_B}{x_1} \underset{\tau_A}{x_2} \ldots \underset{\tau_A}{x_n}$

A-normal form : $x_0 \; x_1 \; x_2 \; \text{——} \; x_n$, $x_0 \in A$, $x_1 \in \tau_A$ ($\tau_B$), $x_2 \in \tau_B$ ($\tau_A$), $x_n \in \tau_A$ ($\tau_A$)

B-normal form : $x_0 \in B$.

(c) $\langle a^m \rangle$ commutes with $\langle a \rangle$
$\langle b^n \rangle$ commutes with $\langle b \rangle$ — generates $G *_A H$.

$\tau_A = \{e, a, \text{—}, a^6\}$

(d) $m = 7$, $n = -6$ : $a^7 = b^{-6}$. $\tau_B = \{e, b, \text{—}, b^5\}$

- $a^{-3} b^2 (ab)^3 b^{-5} = a^{-3} b^2 ababa b^{-4} =$ (central)

$a^{-3} \hookrightarrow a^{-7} \cdot a^4$, $b^{-4} \hookrightarrow b^{-6} \cdot b^2$

$= \cancel{a^{-7}} \cdot \cancel{b^{-6}} \cdot a^4 b^2 ababab^2$.

$b^{-6} \hookrightarrow a^7$, $a^4 \in \tau_A$, $b^2 \in \tau_B$ — — .

- $b^{11} a^{23} b^{-1} a b^{-11} = (b^6 \cdot a^{21} \cdot b^{-6} \cdot b^{-12}) b^5 a^2 b^5 ab =$ (central)

$b^{11} = b^5 \cdot b^6$, $a^{23} = a^{21} \cdot a^2$, $b^{-1} = b^{-6} \cdot b^5$, $b^{-11} = b^{-12} \cdot b$

$a^{21} \hookrightarrow (a^7)^3 = (b^{-6})^3 = b^{-18}$

$= b^{6-18-6-12} = b^{-30}$

$= b^{-30} \cdot b^5 a^2 b^5 ab$.

- $(ab)^{100} = abab \text{———} ab$.