Exercise class 5

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Correction exercise set 4

Exercise 1. Let $m, n \in \mathbb{Z} \setminus \{0\}$ and consider the *Baumslag–Solitar group*

$$BS(m,n) := \langle a, t \mid t^{-1}a^m t = a^n \rangle.$$

(a) Express $BS(m,n)$ as an HNN-extension, find transversals and describe the normal form with respect to these.

(b) Let $m = 7$ and $n = -6$. Find the normal form of the following elements.

- $tata^{-1}ta^2ta^{-2}ta^3ta^{-3}$.
- $t^{-1}at^{-1}a^{-1}t^{-1}a^2t^{-1}a^{-2}t^{-1}a^3t^{-1}a^{-3}$.
- $ta^{100}ta^{-100}t$.

(a) $G = \langle a \rangle \cong \mathbb{Z}$ $\quad A = \langle a^m \rangle \cong m\mathbb{Z} \xrightarrow{\varphi} B = \langle a^n \rangle \cong n\mathbb{Z}$

$mk \longmapsto nk \quad \in \mathbb{Z}$

$a^m \longmapsto a^n \quad \in G$

$G *_\varphi = BS(m,n)$.

$T_A = \{0, 1, \dots, |m|-1\}$ $\quad T_B = \{0, 1, \dots, |n|-1\}$

Normal form: $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} g_n$.

$g_0 \in G = \langle a \rangle$

$\varepsilon_i = -1 \Rightarrow g_i \in T_A \quad (1, a, \dots, a^{|m|-1})$

$\varepsilon_i = 1 \Rightarrow g_i \in T_B \quad (1, a, \dots, a^{|n|-1})$

(b) Just do #2 $\quad t^{-1}a^7t = a^{-6} \Rightarrow$

$$\boxed{\begin{aligned} t^{-1}a^7 &= a^{-6}t^{-1} \\ t^{-1}a^{-7} &= a^6 t^{-1} \end{aligned}}$$

$t^{-1}at^{-1}a^{-1}t^{-1}a^2t^{-1}a^{-2}t^{-1}a^3t^{-1}a^{-3} =$

$\quad t^{-1}a^{-7}a^4 = a^6t^{-1}a^4$

$= t^{-1}at^{-1}a^{-1}t^{-1}a^2t^{-1}a^{-2}t^{-1}a^9t^{-1}a^4 =$

$\quad t^{-1}a^7a^2 = a^{-6}t^{-1}a^2$

$= t^{-1}at^{-1}a^{-1}t^{-1}a^2t^{-1}a^{-8}t^{-1}a^2t^{-1}a^4 =$

$\quad t^{-1}a^{-14}a^6 = a^{12}t^{-1}a^6$

$\checkmark \in T_A \setminus T_B$

$= t^{-1}at^{-1}a^{-1}t^{-1}a^{12}t^{-1}a^6t^{-1}a^2t^{-1}a^4 =$

$\quad t^{-1}a^7a^5 = a^{-6}t^{-1}a^5$

$= t^{-1} a t^{-1} a^{-1} \underbrace{t^{-1} a^{12}}_{t^{-1} a^{7} a^{5} = a^{-6} t^{-1} a^{5}} t^{-1} a^{6} t^{-1} a^{2} t^{-1} a^{4} =$

$= t^{-1} \underbrace{a t^{-1}}_{a^{6} t^{-1}} a^{-7} t^{-1} a^{5} t^{-1} a^{6} t^{-1} a^{2} t^{-1} a^{4} =$

$= \underbrace{t^{-1} a^{7}}_{a^{-6} t^{-1}} t^{-1} t^{-1} a^{5} t^{-1} a^{6} t^{-1} a^{2} t^{-1} a^{4} =$

$= a^{-6} t^{-3} a^{5} t^{-1} a^{6} t^{-1} a^{2} t^{-1} a^{4}.$

$\left(\left\{ \frac{p}{n^k} : p \in \mathbb{Z}, k \in \mathbb{Z} \right\}, + \right) \leq (\mathbb{Q}, +)$

Exercise 2. Show that $BS(1,n) \cong \mathbb{Z}\left[\frac{1}{n}\right] \rtimes_\varphi \mathbb{Z}$, where $\varphi : \mathbb{Z} \to Aut(\mathbb{Z}\left[\frac{1}{n}\right])$ is defined by $\varphi(k)(x) = n^k \cdot x$. The group $BS(1,-1)$ is commonly known as the *Klein bottle group* (why?) and the group $BS(1,1)$ is commonly known as...?

$BS(1,n) = \langle a, t \mid t^{-1} a t = a^{n} \rangle$

$t a t^{-1} \notin \langle a \rangle$

$\mathbb{Z}\left[\frac{1}{n}\right] \rtimes_\varphi \mathbb{Z} \qquad \varphi(k)(x) = n^k \cdot x.$

$(x, k) \cdot (y, \ell) = (x + n^k \cdot y, k + \ell)$

1) $\Theta : BS(1,n) \longrightarrow \mathbb{Z}\left[\frac{1}{n}\right] \rtimes_\varphi \mathbb{Z}$

$a \longmapsto (1, 0)$

$t \longmapsto (0, -1)$

$\Theta(t)^{-1} \Theta(a) \Theta(t) = (0,1)(1,0)(0,-1) = (n, 0) = \Theta(a)^n$

$\Rightarrow \Theta$ is a homomorphism.

2) $\Psi : \mathbb{Z}\left[\frac{1}{n}\right] \rtimes_\varphi \mathbb{Z} \longrightarrow BS(1,n)$

$\left(\frac{m}{n^k}, 0\right) \longmapsto t^{k} a^{m} t^{-k}$

$(0, k) \longmapsto t^{-k}$

$1 \longmapsto a$

$m \longmapsto a^m$

$\frac{m}{n} \longmapsto t a^m t^{-1}$

$\frac{m}{n^k} \longmapsto t^k a^m t^{-k}$

- $\Psi|_{\mathbb{Z}[\frac{1}{n}]}$ is well-defined

$\Psi\left(\frac{m \; n^\ell}{n^k \; n^\ell}\right) = t^{k+\ell} a^{m \cdot n^\ell} t^{-k-\ell} =$

$= t^k \left(t^\ell (a^m)^{n^\ell} t^{-\ell} \right) t^{-k}$

[illegible]

$$= t^{k} \left(t^{l} (a^{m})^{n^{l}} t^{-l} \right) t^{-k}$$

$= a^{m}$ → see * below

$$= t^{k} a^{m} t^{-k} = \Psi\left(\frac{m}{n^{k}}\right)$$

- $\Psi|_{\mathbb{Z}[\frac{1}{n}]}$ is a homomorphism

$$\Psi\left(\frac{m}{n^{k}} + \frac{p}{n^{l}}\right) = \Psi\left(\frac{m n^{l} + p n^{k}}{n^{k+l}}\right) =$$

$$= t^{k+l} a^{m n^{l} + p n^{k}} t^{-k-l} =$$

$$= t^{k} t^{l} (a^{m})^{n^{l}} \left(t^{-l} t^{-k} t^{l} t^{k} \right) (a^{p})^{n^{k}} t^{-k} t^{-l} =$$

(with $t^{l}(a^{m})^{n^{l}}t^{-l} = a^{m}$, $t^{-l} t^{-k} t^{l} t^{k} = 1$, $t^{k}(a^{p})^{n^{k}}t^{-k} = a^{p}$)

$$= t^{k} a^{m} t^{-k} \cdot t^{l} a^{p} t^{-l} = \Psi\left(\frac{m}{n^{k}}\right) \cdot \Psi\left(\frac{p}{n^{l}}\right).$$

- Ψ is a homomorphism

$$\Psi\left(\left(\frac{m}{n^{k}}, s\right) \cdot \left(\frac{p}{n^{l}}, r\right)\right) = \Psi\left(\frac{m}{n^{k}} + \frac{n^{s} p}{n^{l}}, s+r\right) =$$

$$\overset{\text{def of } \Psi}{=} \Psi\left(\frac{m}{n^{k}} + \frac{n^{s} p}{n^{l}}, 0\right) \Psi(0, s+r) \overset{\Psi|_{\mathbb{Z}[\frac{1}{n}]},\ \Psi|_{\mathbb{Z}} \text{ is a hom}}{=}$$

$$= t^{k} a^{m} t^{-k} \cdot t^{l} a^{p \cdot n^{s}} t^{-l} \cdot t^{-s-r} =$$

$$= \left(t^{k} a^{m} t^{-k} \right) \cdot t^{l} \left(t^{-s} a^{p} t^{s} \right) t^{-l} t^{-s} t^{-r} =$$

$$= \left[t^{k} a^{m} t^{-k} \cdot t^{-s} \right] \left[t^{l} a^{p} t^{-l} \cdot t^{-r} \right] =$$

$$= \Psi\left(\frac{m}{n^{k}}, s\right) \cdot \Psi\left(\frac{p}{n^{l}}, r\right).$$

Check: Ψ and Θ are mutually inverse $\Rightarrow$ isomorphisms.

$$BS(1,1) = \langle a, t \mid t^{-1} a t = a \rangle = \mathbb{Z}^{2}$$

$BS(1,-1) = \langle a, t \mid t^{-1} a t = a^{-1} \rangle = \pi_1(\text{Klein bottle})$

Proposition (Moldavanskii, 1991). Let $m, n, m', n' \in \mathbb{Z} \setminus \{0\}$. Then $BS(m,n) \cong BS(m',n')$ if and only if $(m',n') \in \{(m,n), (-m,-n), (n,m), (-n,-m)\}$.

Exercise 3. Prove $\Leftarrow$ of the proposition above. Prove $\Rightarrow$ in case $(m,n) = (1,1)$.

$\Leftarrow$ $BS(m,n) = \langle a, t \mid t^{-1} a^m t = a^n \rangle$

$BS(-m,-n) \overset{b = a^{-1}}{=} \langle b, t \mid t^{-1} b^{-m} t = b^{-n} \rangle$

$BS(n,m) \overset{s = t^{-1}}{=} \langle a, s \mid a^m = s^{-1} a^n s \rangle$

$\Rightarrow$ Claim: $BS(m,n)$ is non-abelian for all other (m,n).

Case 1: $m \neq n$. $a^m \overset{\text{abelian}}{=} t^{-1} a^m t \overset{\text{pres}}{=} a^n$.

But a has ∞ order. ↯

Case 2: $m = n$, $|n| > 1$. Then $T_A = \{1, -, a^{|n|-1}\} = T_B$

$\rightsquigarrow$ normal forms

$[a^{-1}, t] = a^{-1} t a t^{-1}$ in normal form $\Rightarrow \neq 1$.

$a^{-1} = g_0$, $t^{\varepsilon_1} g_1 t^{\varepsilon_2}$

So not abelian.

Definition. A group G is *Hopfian* if every surjective homomorphism $G \to G$ is injective.

Exercise 4. Let again $BS(2,3) = \langle a, t \mid t^{-1} a^2 t = a^3 \rangle$. Set $\eta(a) = a^2$ and $\eta(t) = t$.

(a) Show that η defines a homomorphism $BS(2,3) \to BS(2,3)$.

(b) Find an element that maps to a, and deduce that η is surjective.

(c) Use this to find a non-trivial element in the kernel, and deduce that η is not injective.

(a) $\eta(t)^{-1} \eta(a)^2 \eta(t) = t^{-1} a^4 t = (t^{-1} a^2 t)^2 = (a^3)^2 = (a^2)^3 = \eta(a)^3$

(b) If $\exists x : \eta(x) = a$, then $\operatorname{im} \eta \supset \langle \eta(t), \eta(x) \rangle = \langle t, a \rangle = BS(2,3)$

$\eta(t^{-1}at) = t^{-1}a^2t = a^3 \Rightarrow \eta(a^{-1}\cdot t^{-1}at) = a^{-2}\cdot a^3 = a$

(c) $\eta(x^2) = a^2 = \eta(a) \Rightarrow a^{-1}\cdot x^2 \in \ker \eta$.

$a^{-1}\, a^{-1}t^{-1}at\,a^{-1}t^{-1}at = a^{-2}t^{-1}a^{-1}ta^2t^{-1}at =$

$ta^{-3}a^2 = a^{-2}ta^2 \qquad t^{-1}a^{-2}\cdot a = a^{-3}t^{-1}a$

$= a^{-5}t^{-1}at^{-1}a^2t^{-1}at$ NORMAL FORM !

$\Rightarrow a^{-1}x^2 \neq 1.$

Exercise 5. Consider the *Higman group*

$$G := \langle a,b,c,d \mid bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2\rangle.$$

Denote by $G_{ab} := \langle a,b \mid bab^{-1} = a^2\rangle$ (BS(1,2), ex. 2) $\cong \mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}$, and similarly G_{bc}, G_{cd}, G_{da}.

(a) Let $G_{abc} := G_{ab} *_{\langle b\rangle} G_{bc}$. Show that a and c generate freely a subgroup $F \leq G_{abc}$.

(b) Define similarly G_{cda}. Show that $G \cong G_{abc} *_F G_{cda}$ and deduce that G is infinite.

(a) $G_{abc} = \langle a,b,c \mid bab^{-1} = a^2, cbc^{-1} = b^2\rangle$, $F := \langle a,c\rangle \leq G$.

$\{a^i : i\in\mathbb{Z}\}$ all in different $\langle b\rangle$-cosets of G_{ab}: $b^k a^i = b^l a^j \Leftrightarrow$
$\Leftrightarrow k=l,\ i=j$ (HNN)

Similarly $\{c^j : j\in\mathbb{Z}\}$ " of G_{bc}.

So can extend to a system T_A of reps of $G_{ab}/\langle b\rangle$, T_c of $G_{bc}/\langle b\rangle$ $\rightsquigarrow$ normal forms in G_{abc}

Then a reduced product $a^{n_1}c^{m_1} \text{—} a^{n_k}c^{m_k}$ is in normal form.
So unique $\Rightarrow \langle a,c\rangle$ is free.

(b) $G \cong G_{abc} *_F G_{cda}$ ✓

G contains $G_{ab} \leq G_{abc}$, F, ... all infinite.
$\cong \mathbb{Z}[\frac{1}{2}]\rtimes\mathbb{Z}$

Exercise 6. Use the following two steps to show that all finite quotients of G are trivial.

(a) Show that G admits an automorphism permuting cyclically a, b, c, d. Deduce that if G has a non-trivial finite quotient, then it also has one in which a, b, c, d all have the same order.

(b) Show that in a finite group, if g is conjugate to g^2 by an element of the same order, then $g = 1$.

(a)+(b) ⇒ 6 : Say G has a finite quotient $Q \neq 1$. (a) ⇒ we may assume that the images of a, b, c, d in Q all have the same order.

Then $bab^{-1} = a^2$, $o(a) = o(b)$ so (b) ⇒ $a = 1$
⇒ $b = c = d = 1$ ↯

(a) $\alpha : G \longrightarrow G : a \mapsto b \mapsto c \mapsto d \mapsto a$.

$\alpha(b)\alpha(a)\alpha(b)^{-1} = cbc^{-1} = b^2 = \alpha(a)^2$, etc. ...

α is a hom, $\alpha \circ \alpha \circ \alpha \circ \alpha = id_G$ ⇒ $\alpha \in Aut(G)$.

Suppose G has a finite index normal subgroup N ($\neq G$) : G/N fin. quotient ($\neq \{1\}$)

$M := \bigcap_{i=0}^{3} \alpha^i(N) = \bigcap_{i \in \mathbb{Z}} \alpha^i(N)$

Then M is finite-index (finite intersection), normal, α-invariant

$G/M \neq \{1\}$ is a finite quotient

α induces $\alpha : G/M \to G/M$ permuting a, b, c, d cyclically.
Automorphism ⇒ $o(a) = o(b) = o(c) = o(d)$.

$*$: An intersection of two finite-index subgroups H, K has finite index

Pf: $G \curvearrowright (G/H \times G/K)$. $\text{Stab}(eH, eK) = H \cap K$

Orbit-stab : $[G : H \cap K] = |G \cdot (eH, eK)| \leq |G/H \times G/K| < \infty$.

(b) $n = o(g) = o(h)$, $hgh^{-1} = g^2$. $g = h^n g h^{-n} \overset{*}{=} g^{2^n}$.

$g^{2^n - 1} = 1$ ... $n \mid 2^n - 1$ (order) ... $2^n \equiv 1 \mod n$

$\Rightarrow \quad g^{2^n - 1} = 1 \quad \overset{\text{order}}{\Longrightarrow} \quad \boxed{n \mid 2^n - 1}$, i.e. $2^n \equiv 1 \mod n$.

↪ impossible for $n \neq 1$

Let p be the smallest prime dividing n, assuming $n \neq 1$.

$2^n \equiv 1 \mod p \implies p \neq 2$.

$2^{p-1} \equiv 1 \mod p$ ← Fermat's little thm (because $\mathbb{F}_p^\times$ has order $p-1$)

$\Rightarrow$ the order of 2 in $\mathbb{F}_p^\times$ divides n, $p-1$.

$\Rightarrow$ $(n, p-1)$ have a common prime divisor

$\Rightarrow$ $\exists$ a prime divisor of n, smaller than p ↯

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$h g h^{-1} = g^2 \qquad h^2 g h^{-2} = h g^2 h^{-1} = (h g h^{-1})^2 = (g^2)^2 = g^4 \ldots$

$h^n g h^{-n} = h g^{2^{n-1}} h^{-1} = (g^{2^{n-1}})^2 = g^{2^n}$