Exercise class 6

Donnerstag, 13. Mai 2021   12:03

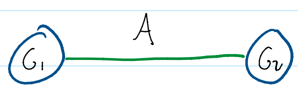
Correction exercise set 5

Exercise 1. Let $m, n \in \mathbb{Z} \setminus \{0\}$ and consider the torus knot group $K_{m,n} = \langle a, b \mid a^m = b^n \rangle$ (cf. Exercise set 3). Describe the tree obtained via Theorem 6.12: define it, draw a picture, and compute the degrees of its vertices.

↳ Bass–Serre tree

$K_{m,n} = G_1 *_A G_2$ where $G_1 = \langle a \rangle \cong \mathbb{Z}$, $G_2 = \langle b \rangle \cong \mathbb{Z}$

$a^m \hookleftarrow A \hookrightarrow b^n$

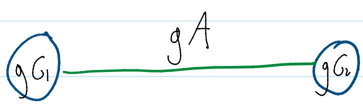


$X^0 = G/G_1 \sqcup G/G_2$

$X^1_+ = G/A$

$\boxed{\text{recall: } G_1 \text{ will be the stabilizer of a vertex so } G/G_1 \cong \text{corresponding orbit. Same for } A}$

Adjacency:

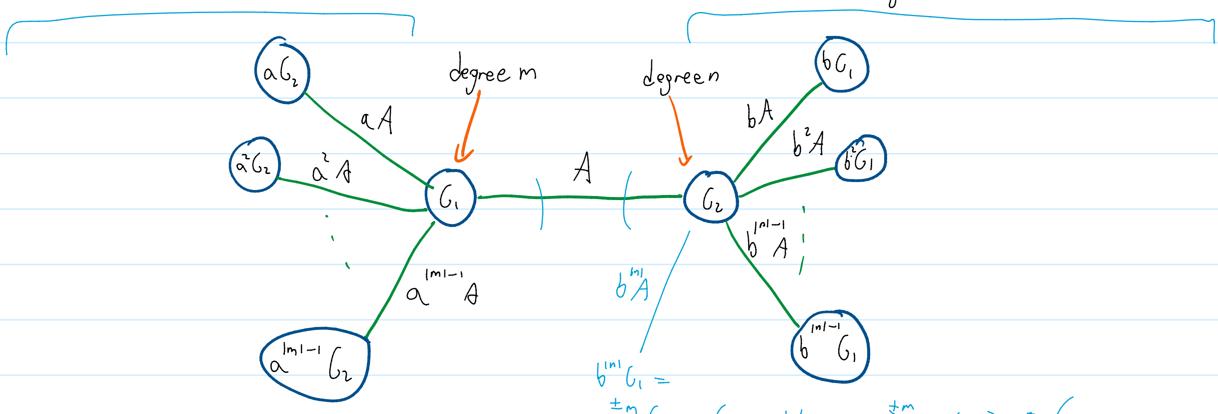


$g\langle a \rangle : g\langle b \rangle = \langle b \rangle$

$gG_1 : gG_2 = G_2$

$\rightsquigarrow g \in \langle b \rangle$

$gG_2 : gG_1 = G_1 \rightsquigarrow g \in \langle a \rangle$



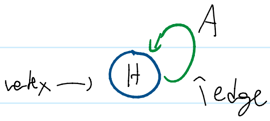
Each "side") (is a bi-regular tree $(m-1, n-1)$

Exercise 2. Let $m, n \in \mathbb{Z} \setminus \{0\}$ and consider the Baumslag–Solitar group $BS(m, n) = \langle a, t \mid t^{-1}a^m t = a^n \rangle$ (cf. Exercise set 4). Describe the tree obtained via Theorem 7.14: define it, draw a picture, and compute the degrees of its vertices.

Bass–Serre tree

$G = BS(m,n) = H *_\varphi$ where $H = \langle a \rangle$, $A = \langle a^m \rangle$,

$\varphi : A \to H : a^m \mapsto a^n$.



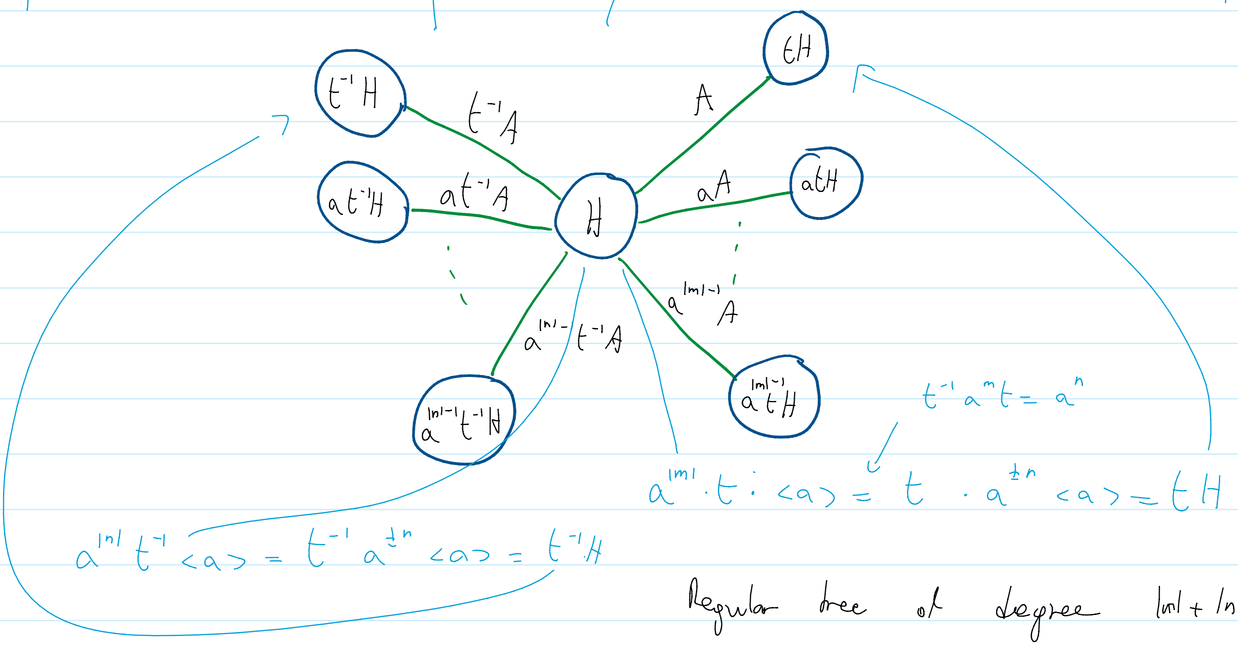
$\boxed{\text{recall: } H \text{ will be the stabilizer of a vertex so } G/H \cong \text{corresponding orbit. Same for } A}$

$X^0 = G/H$

$X^1_+ = G/A$

Adjacency: $gH \overset{gA}{\text{———}} gtH$

$\underbrace{gH : \ gtH = H \rightsquigarrow g \in Ht^{-1} = \langle a \rangle t^{-1}}$ $\qquad$ $\underbrace{gtH : \ gH = H \rightsquigarrow g \in H = \langle a \rangle}$



Regular tree of degree $|m| + |n|$.

Exercise 3. Let σ, τ be commuting automorphisms of a tree X acting without inversion.

(a) Show that σ preserves $\mathring{\tau}$ or $\vec{\tau}$ (depending on whether τ is a rotation or a translation).

(b) Deduce that $\langle \sigma, \tau \rangle$ has a global fixed point or preserves a line in X.

$\min(\tau) = \{ x \in X^0 \text{ s.t. } d(x, \tau(x)) = |\tau| \}$. (minimal possible)

They are the vertices of a tree: $\mathring{\tau}$ if $|\tau| = 0$

$\vec{\tau}$ (a line) if $|\tau| > 0$.

(a) $\sigma(\min(\tau)) = \sigma\{x : d(x, \tau(x)) = |\tau|\} =$

$= \{\sigma(x) : d(x, \tau(x)) = |\tau|\} =$ $\quad y = \sigma(x)$

$= \{y : d(\sigma^{-1}(y), \tau\sigma^{-1}(y)) = |\tau|\} =$ $\quad \sigma$ is an isometry

$= \{\sigma(x) : d(x, \tau(x)) = |\tau|$ $=$ $y = \sigma(x)$

$= \{y : d(\sigma^{-1}(y), \tau\sigma^{-1}(y)) = |\tau|\}$ $=$ σ is an isometry

$= \{y : d(y, \sigma\tau\sigma^{-1}(y)) = |\tau|\} = \min(\tau)$.

$\sigma\tau\sigma^{-1} = \tau$ b/c commute

$\leadsto \sigma(\mathring{\tau}/\vec{\tau}) = \mathring{\tau}/\vec{\tau}$.

(b) Case 1: τ is a translation $\leadsto$ τ and σ preserve the line $\vec{\tau}$.

Case 2: τ is a rotation $\leadsto$ σ preserves $\mathring{\tau}$, a tree. Then look at $\sigma|_{\mathring{\tau}} \curvearrowright \mathring{\tau}$.
Here, σ either fixes a point p or translates along a line L $\leadsto$ this is fixed by τ

Exercise 4. Let A be a finitely generated abelian group acting without inversion on a tree X.

(a) Show that A has a global fixed point or preserves a line in X.

(b) Deduce that if the quotient $A\backslash X$ is finite, then ~~either X is finite~~ or there exists a line L (respectively, a point p) and an integer $d \geq 1$ such that every vertex in X is at distance at most d from a vertex in L (respectively, from p).

(c) Generalize the previous item to finitely generated virtually abelian groups (i.e., groups with an abelian subgroup of finite index).

(a) Same proof: $A = \langle \sigma_1, \ldots, \sigma_n \rangle$.

Case 1: σ_1 is a translation $\leadsto$ A preserves $\vec{\sigma_1}$ by ex. 3.

Case 2: σ_1 is a rotation $\leadsto$ apply induction on $\langle \sigma_2, \ldots, \sigma_n \rangle|_{\mathring{\sigma}_1}$.

(b) Say A has a global fixpoint p. Then $\forall\ a \in A$,
$d(v, p) = d(a(v), a(p)) = d(a(v), p)$.
$A \curvearrowright$ by isometries $\qquad a(p) = p$

So two vertices in the same A-orbit have the same distance from p.

$A \backslash X$ is finite $\rightsquigarrow$ finitely many orbits $\rightsquigarrow$ $d(v, p)$ is bounded.

Same with the invariant line $\left(\text{look at } d(v, L) = \min \{ d(v, \ell) : \ell \in L \} \right)$.

(c) Suppose $G \curvearrowright X$ w/o inversion and $\exists$ $A \leq G$ f.g. abelian : $[G : A] < \infty$.

Claim: If $|G \backslash X| < \infty$, then X is in a bdd nbh of a point / line.

To prove the claim it suffices to show that $|G \backslash X| < \infty \Rightarrow |A \backslash X| < \infty$

Pf. $G \backslash X$ finite means: finitely many orbits of vertices, and edges.

Let T be a finite set of representatives of $A \backslash G$. Then:

$$\underbrace{G \cdot \underset{X^0}{\overset{\in}{v}}}_{\text{one } G\text{-orbit}} = \left(\coprod_{t \in T} A \cdot t \right) \cdot v = \underbrace{\bigcup_{t \in T} A \cdot (t(v))}_{\leq |T| \ A\text{-orbits}}$$

So A also has fin. many orbits of vertices.
Same w/ edges. $\square$

RmK: Turns out: if G is f.g., any finite-index subgroup is also f.g.

Proposition (Moldavanskii, 1991). Let $m, n, m', n' \in \mathbb{Z} \setminus \{0\}$. Then $BS(m, n) \cong BS(m', n')$ if and only if $(m', n') \in \{(m, n), (-m, -n), (n, m), (-n, -m)\}$.

Exercise 5. Show that the Klein bottle group is virtually abelian. Then use Exercises 2 and 4 to prove $\Rightarrow$ of Moldavanskii's Proposition in case $(m, n) = (1, -1)$.

$BS(1, -1) = \langle a, t \mid t^{-1} a t = a^{-1} \rangle = G$.

$A = \langle a, t^2 \rangle$: $t^{-2} a t^2 = t^{-1} (t^{-1} a t) t = t^{-1} a^{-1} t = (t^{-1} a t)^{-1} = a$.

$\rightsquigarrow A$ is abelian

$[G : A] = 2$ because any element of G is written $a^i t^j$

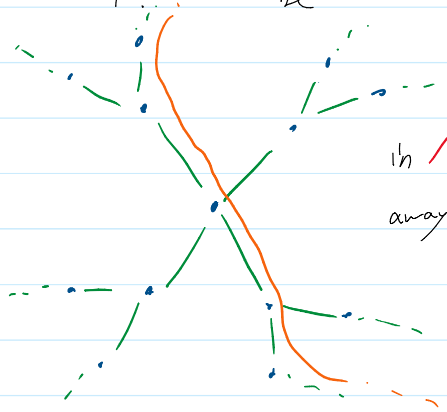
Suppose $G = BS(1, -1) \cong BS(m, n)$. WTS $m = -n = \pm 1$.

Suppose $G = BS(1,-1) \cong BS(m,n)$. WTS $m = -n = \pm 1$.

By exercise 2, $\curvearrowright X$ an $(|m|+|n|)$-regular tree.

By exercise 4, X is in the bdd nbh of a point / line.

This implies that $|m| = |n| = 1$. Otherwise X is $(d \geq 3)$-regular, and X



in direction, you get as far away from as you want.

Then $m \neq n$ by exercise 3 in exercise set 4.

$\rightsquigarrow$ $m = -n = \pm 1$.

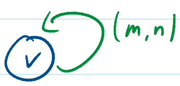
$t_e = 1$, $e \in E(T)$

Definition. Let Y be a finite connected graph with a spanning tree T, and a *weight function*, i.e. a map $w : E(Y) \to \mathbb{Z} \setminus \{0\}$. Let G be a group generated by elements $\{g_v : v \in V(Y), t_e : e \in E(Y) \setminus \cancel{E(T)}\}$ subject to the relations $\{t_e t_{\overline{e}} = 1, t_e g_{\alpha(e)}^{w(\overline{e})} t_e^{-1} = g_{\omega(e)}^{w(e)}\}$. Then G is called the *GBS group* corresponding to the weighted graph (Y, w).

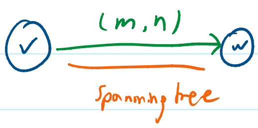
Exercise 6. Show that GBS groups act on trees without inversion with finite quotient and all vertex and edge stabilizers infinite cyclic.

Remark. Next week we will see that the converse also holds: all groups admitting such actions are GBS groups.

EX:



$\rightsquigarrow$ $BS(m,n) = \langle g_v, t_e \mid t_e g_v^m t_e^{-1} = g_v^n \rangle$



$\rightsquigarrow$ $K_{m,n} = \langle g_v, g_w \mid g_v^m = g_w^n \rangle$.

A GBS group is the fundamental group of a finite, connected gog with vertex groups $\langle g_v \rangle \cong \mathbb{Z}$; and edge groups with edge inclusions given by the weights.

Weights $\neq 0$ $\rightsquigarrow$ edge groups also $\cong \mathbb{Z}$.

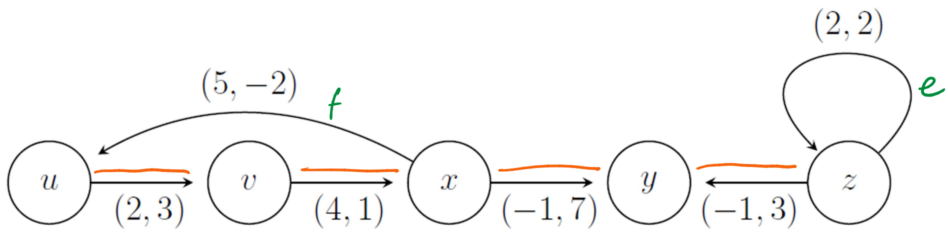
w/ finite quotient

weights $\neq 0$ $\rightsquigarrow$ edge groups also $\cong \mathbb{Z}$,

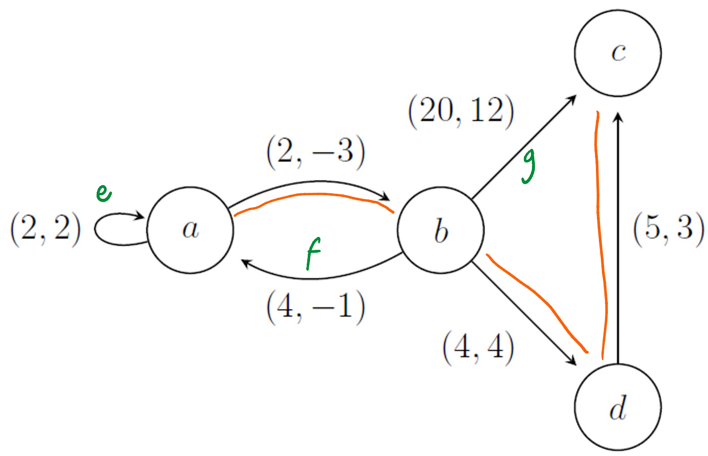
Then by fundamental thm $\exists$ a tree on which it acts w/ finite quotient w/ all vertex / edge stabilizers conjugate to the vertex / edge groups $\rightsquigarrow$
$\rightsquigarrow$ infinite cyclic.

Converse holds by converse of fundamental thm.

Exercise 7. Write down presentations of the GBS groups corresponding to the following weighted graphs. Below, an orientation of each edge is chosen, and the oriented edge e is labeled by $(w(\overline{e}), w(e))$.



$$G = \langle u, v, x, y, z, e, f \mid u^2 = v^3,\ v^4 = x,\ x^{-1} = y^7,\ z^{-1} = y^3,\ e z^2 e^{-1} = z^2,\ f x^5 f^{-1} = u^{-2} \rangle$$



$$G = \langle a, b, c, d, e, f, g \mid a^2 = b^{-3},\ b^4 = d^4,\ d^5 = c^3,\ e a^2 e^{-1} = a^2,\ f b^4 f^{-1} = a^{-1},\ g b^{20} g^{-1} = c^{12} \rangle$$