Exercise class 7

Donnerstag, 27. Mai 2021 10:37

Correction exercise set 6

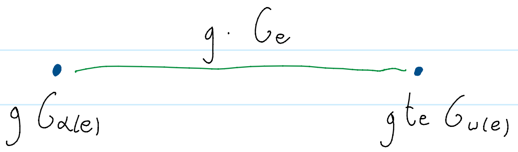
Exercise 1. Let G be a free product with amalgamation (respectively, an HNN extension). Express G as the fundamental group of a graph of groups, and show that its Bass–Serre tree coincides with the tree obtained via Theorem 6.12 (respectively, Theorem 7.14).

– $(\mathbb{G}, Y)$ gog $\overset{\text{Bass-Serre tree}}{\rightsquigarrow}$ tree X on which $\underline{G = \pi_1(\mathbb{G}, Y)}$ acts w/o inversion, all vertex / edge stabilizers conjugate to vertex / edge groups of $(\mathbb{G}, Y)$.

- $X^0 = \coprod_{v \in Y^0} G/G_v$; $X^1_+ = \coprod_{e \in Y^1_+} G/G_e$.

a given spanning tree ↓

(recall: $t_e = 1 \iff e \in T$)

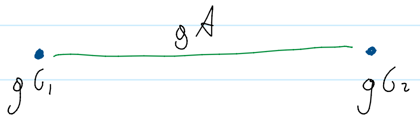


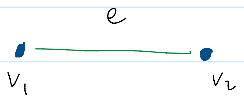
1) $G = G_1 *_A G_2 \overset{\text{THM 6.12}}{\rightsquigarrow}$ tree $X^0 = G/G_1 \amalg G/G_2$

$X^1_+ = G/A$.

$\quad ||$

$\pi_1(\mathbb{G}, Y)$





$t_e = 1$ since $\{e\}$ is a spanning tree.

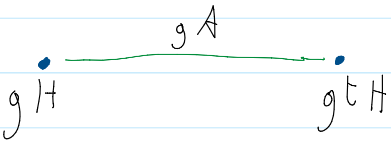
$G_{v_1} = G_1$, $G_{v_2} = G_2$, $G_e = A$, given inclusion $A \hookrightarrow G_1, G_2$.

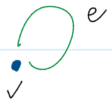
2) $G = H *_\varphi$ where
$$\begin{tikzcd} \varphi: A \arrow[rr, "\cong"] \arrow[dr, phantom, "\geq"] & & B \arrow[dl, phantom, "\leq"] \\ & H & \end{tikzcd}$$
$\overset{\text{THM 7.14}}{\rightsquigarrow}$ tree $X^0 = G/H$

$X^1 = G/A$

$\quad ||$

$\pi_1(\mathbb{G}, Y)$

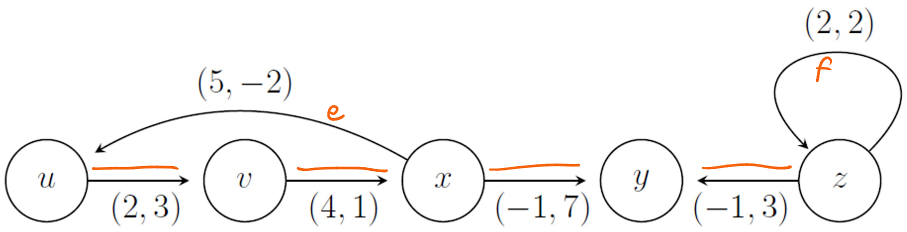




$G_v = H$, $G_e = A$, $t_e = t \neq 1$

$\iota_{\bar{e}} : A \hookrightarrow H$, $\quad \iota_e : A \xrightarrow{\varphi} B \hookrightarrow H$

Exercise 2. Consider the two GBS groups from Exercise 7 in Exercise set 5. Describe their Bass–Serre trees: define them, draw pictures, and compute the degrees of their vertices.



$G_u = \langle u \rangle \cong \mathbb{Z}$

$G_e = \langle e \rangle \cong \mathbb{Z}$

$u \xrightarrow{(2,3)} v$

$\rightsquigarrow$

$\iota_{\bar{e}}(e) = u^2$

$\iota_e(e) = v^3$

i.e., $t_e^{-1} u^2 t_e = v^3$.

General result: $\underset{v}{\bullet} \overset{e}{\underset{(m,n)}{\text{———}}} \underset{w}{\bullet} \subseteq Y$ leads to (?) in the Bass-Serre tree!

The corr. edges in X are: $\underset{gG_v}{\bullet} \overset{gG_e}{\text{———}} \underset{g t_e G_w}{\bullet}$ $\quad t_e^{-1} v^m t_e = w^n$.

→ Fix $gG_v = G_v$ $\bullet$ ——— $\bullet$ $g t_e G_w$ where $g \in \{1, v, v^2, \dots, v^{|m|-1}\}$

$\iff g \in G_v$

(Then repeats since $v^{|m|} t_e G_w = t_e w^{\pm n} G_w = t_e G_w$)

$g G_v$ where $g \in \{t_e^{-1}, w t_e^{-1}, \dots, w^{|n|-1} t_e^{-1}\}$ $\bullet$ ——— $\bullet$ Fix $g t_e G_w = G_w$ ←

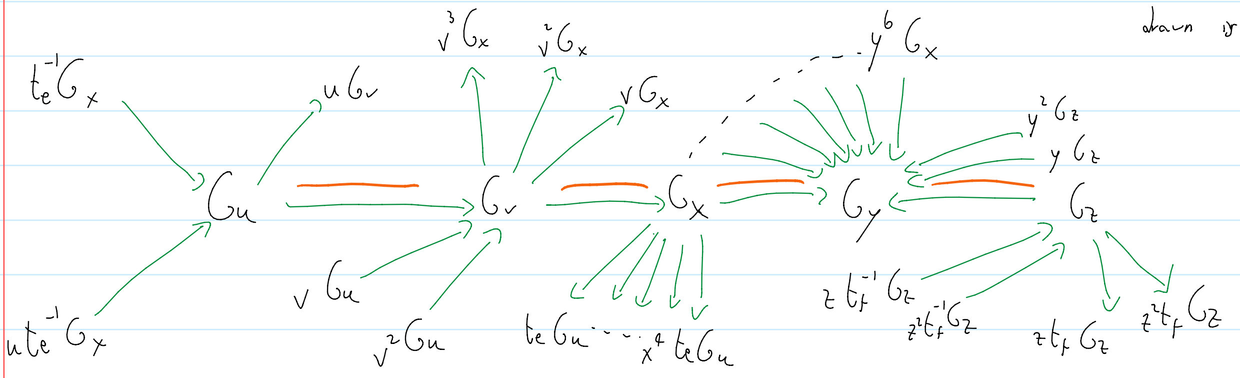
$\iff g \in G_w t_e^{-1}$

(Then repeats since $w^{|n|} t_e^{-1} G_v = t_e^{-1} v^{\pm n} G_v = t_e^{-1} G_v$)

Upshot: This edge gives you m edges gG_e outgoing from G_v,

n " incoming to G_w.

/ = lift of spanning tree drawn is its 1-nbh



Note: The degree of G_v is the sum of abs values of outgoing/incoming

Note: The degree of G_v is the sum of abs values of outgoing / incoming edge weights. Eg degree of $G_u = 2+2 = 4$.

Proposition (Bass–Serre). Let G be the fundamental group of a finite connected graph of groups with finite vertex groups. Then G is virtually free.

Exercise 3. Let H be a normal subgroup of G that intersects trivially each vertex group. Show that H is free.

$H \cap G_v = \{1\}$ $\forall v$ $\overset{\text{normal}}{\rightsquigarrow}$ $H \cap gG_vg^{-1} = \{1\}$ $\forall g \in G$ too.

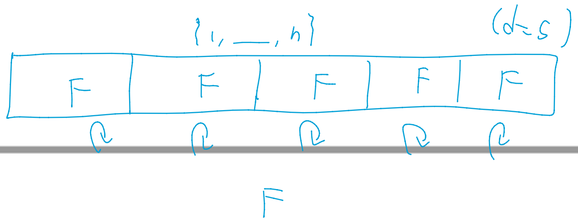
Now let $G \curvearrowright X$ Bass-Serre tree.

Restricting to $H \curvearrowright X$,

is free! B/c any vertex stab. of $G \curvearrowright X$ is of the form gG_vg^{-1}.

By thm 4.6, H is free.

Exercise 4. Let F be a finite group and $n = d \cdot |F|$ for some $d \geq 1$. We can embed F into S_n as follows: choose injective maps $\iota_1, \ldots, \iota_d : F \to \{1, \ldots, n\}$ with disjoint image, and let F act on $\iota_i(F)$ by $g \cdot \iota_i(x) = \iota_i(gx)$. We call such an embedding *standard*.



(a) Show that there is a correspondence between standard embeddings of F into S_n and free actions of F on $\{1, \ldots, n\}$.

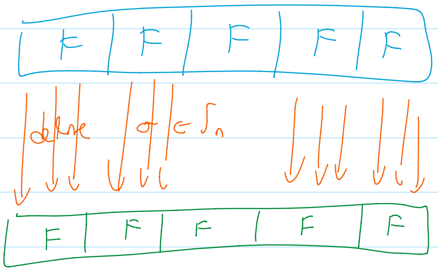
(b) Deduce that any two standard embeddings of F into S_n are conjugate.

(c) Deduce that any standard embedding of a subgroup of F into S_n can be extended to a standard embedding of F into S_n.

$\overset{\vee|}{E}$

(a) $F \curvearrowright \{1, \ldots, n\}$ free, then for each orbit O, $F_{/\{1\}} \overset{\text{orbit-stabilizer}}{\cong} O$ as F-sets.

(b)



: $\iota_1, \dots, \iota_d$

: $j_1, \dots, j_d$

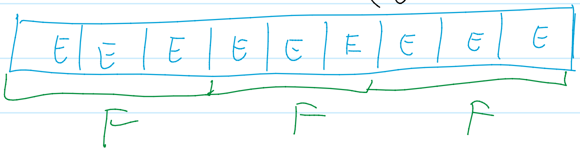
$\sigma(\iota_i(x)) := j_i(x)$.

Then $(\sigma g \sigma^{-1}) \overset{\iota\text{-action}}{\cdot} j_i(x) =$

$= \sigma g \cdot \iota_i(x) = \sigma \cdot \iota_i(gx) \overset{\text{like } j\text{-action}}{=} j_i(gx)$

(↙ $[F:E]=3$)

(c)



($E \curvearrowright F$ is free!)

Let's assume $n = |F|$ (for larger ⇝ disjoint unions)

$\{1, \dots, n\} = \coprod_{i=1}^{d} \iota_i(E)$ $\quad d = n/|E| = [F:E]$.

$F = \coprod_{i=1}^{d} E t_i$ coset reps t_i.

$\iota : F \hookrightarrow \{1, \dots, n\} : h \cdot t_i \longmapsto \iota_i(h)$

Then ι gives a standard embedding $F \hookrightarrow S_n$. For $h \in E$

$h \overset{F\text{-action}}{\cdot} \iota(h' \cdot t_i) = \iota(hh' \cdot t_i) = \iota_i(hh') = h \overset{E\text{-action}}{\cdot} \iota_i(h') =$

$= h \cdot \iota(h' \cdot t_i)$

So it is an extension!

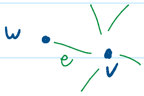
Exercise 5. Let G be as in the proposition, and let n be a multiple of the order of all vertex groups. Use Exercise 4 to define a homomorphism $G \to S_n$ such that the restriction to each vertex group is a standard embedding, and show that the kernel satisfies the hypotheses of Exercise 3.

uses that $|Y| < \infty$

Hint. Start by assuming that the underlying graph is a tree: for this you only need (c).

1) Assume $G = \pi_1(\mathcal{G}, Y)$, $Y =$ tree.

Take $v \in Y$, define a standard embedding $G_v \hookrightarrow S_n$. $(|G_v| \mid n)$



This defines a st. embedding on $G_e \subseteq G_v$

Using (c), extend it to a " on G_w.

Do this ∀ neighbours of v, then induction ⇝ all of Y.

possible b/c Y is a tree!

This gives st. emb. $G_v \longrightarrow S_n$ $\forall$ v s.t.
both restrictions to G_e coincide $\rightsquigarrow$ extends to a hom $\varphi : G \longrightarrow S_n$

$\dot{v} \;\underset{e}{\overline{\qquad}}\; \dot{w}$

$$\langle G_v \mid G_v \text{ and } G_w \text{ are amalg. along } G_e \rangle$$

spanning tree.

2) More generally, $G = \pi_1(\mathcal{G}, Y, T)$

Use 1) pretending $Y = T$ $\rightsquigarrow$ defines st. emb. $G_v \longrightarrow S_n$ $\forall v$
s.t. relations at edges in T are satisfied.

Need to define $\varphi(t_e) \in S_n$ $\forall$ $e \notin T$, s.t.

$\dot{v} \;\overset{\xrightarrow{\iota_e}}{\underset{\underset{\iota_{\bar e}}{\longleftarrow}}{\underset{e}{\overline{\qquad}}}}\; \dot{w}$

$$\varphi(t_e)^{-1} \varphi\big(\underbrace{\iota_{\bar e}(\overset{\Uparrow}{\underset{G_e}{x}})}_{\in G_v}\big) \varphi(t_e) = \varphi\big(\underbrace{\iota_e(x)}_{\underset{G_w}{\Uparrow}}\big)$$

But now

$$\begin{tikzcd} & G_v \arrow[r, "\varphi"] & S_n \\ G_e \arrow[ur, "\iota_{\bar e}"] \arrow[dr, "\iota_e"'] & & \\ & G_w \arrow[r, "\varphi"'] & S_n \end{tikzcd}$$

two standard embeddings of G_e $\overset{(b)}{\rightsquigarrow}$ conjugate by some $\sigma \in S_n$

Define $\varphi(t_e) = \sigma$ $\rightsquigarrow$ done!

3) Let $H := \operatorname{Ker} \varphi$. Then $H \cap G_v$ acts trivially on $\{1, \ldots, n\}$.
But G_v acts freely on $\{1, \ldots, n\}$

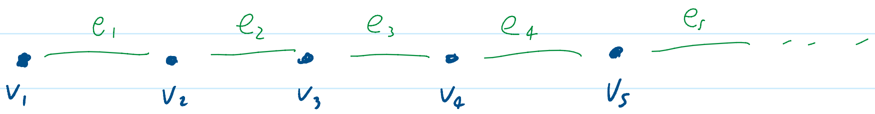
$\rightsquigarrow$ $H \cap G_v = \{1\}$, + H is normal + finite-index
$\overset{3}{\Longrightarrow}$ H is free

Exercise 6. Let G be a *locally finite group*, that is, a group such that every finitely generated subgroup is finite. Suppose moreover that G is countably infinite. Show that G is the fundamental group of a connected graph of groups with finite vertex groups, but it is not virtually free.

Examples of such groups include the group S_∞ of finitely supported permutations of $\mathbb{N}$, and the rational subgroup $\mathbb{Q}/\mathbb{Z}$ of the circle group.

- $G = \{g_1, g_2, \ldots\}$ (countable). $G_i = \langle g_1, \ldots, g_i \rangle$: finite.

Then $G = \bigcup_{i \geq 1} G_i$, $G_i \overset{\iota_i}{\hookrightarrow} G_{i+1}$.



$G_{v_i} = G_i$, $\qquad G_{e_i} = G_{v_i}$,

$$\begin{tikzcd} \iota_{\bar{e}_i} : G_{e_i} \arrow[r] \arrow[d, phantom, "\shortparallel"] & G_{v_i} \\ \mathrm{id} : G_i \arrow[r] & G_i \end{tikzcd}$$

Then $G = \pi_1(\mathcal{G}, Y, \underset{\text{spanning tree}}{Y}) =$

$= \langle G_i \mid G_i \overset{\iota}{\subseteq} G_{i+1} \rangle$

$$\begin{tikzcd} \iota_{e_i} : G_{e_i} \arrow[r] \arrow[d, phantom, "\shortparallel"] & G_{v_{i+1}} \\ \iota_i : G_i \arrow[r, hook] & G_{i+1} \end{tikzcd}$$

$= \bigcup G_i = G.$

But: G is not virtually free: it is torsion so it cannot contain a non-trivial $\underbrace{\text{free subgroup}}_{\text{torsion-free}}$ (ex. #2)

Proposition (Scott)**.** Let F be a finitely generated free group, α an automorphism of F and H a finitely generated subgroup of F. If $\alpha(H) \subseteq H$, then $\alpha(H) = H$.

We prove the proposition in case H is a *free factor* of F: namely, there exists another subgroup K such that $F = H * K$ (actually this special case is a step towards the proof of the general case, which uses some more advanced tools).

Exercise 7. Let F, α, H be as in the proposition, and assume that H is a free factor of F.

(a) Suppose that $H' \leq H$ is another free factor of F. Show that H' is a free factor of H.

(b) Apply this to $H' = \alpha(H)$, and compare the ranks to conclude.

THM (Kurosh for free products): $G \leq A * B$. Then $\exists$ a free group $E \leq A * B$ and systems T_A of reps of double cosets $G \backslash A * B / A$
T_B " $G \backslash A * B / B$

s.t. $G = E * \left(\mathop{*}_{t \in T_A} G \cap tAt^{-1} \right) * \left(\mathop{*}_{t \in T_B} G \cap tBt^{-1} \right)$

In particular, take $t \in T_A$ representing $G \cdot 1 \cdot A$ (so $t \in G \cdot A$)

then $\boxed{G \cap tAt^{-1}}$ is a free factor of G.

(a) Apply the above to $H \leq H' * K' = F$.

Then $\exists \; t \in H \cdot H' \overset{H' \leq H}{=} H$ s.t. $H \cap tH't^{-1} \overset{H' \leq H}{=} tH't^{-1}$ is a free factor of H.

$\rightsquigarrow$ H' is a free factor of $t^{-1}Ht \overset{t \in H}{=} H$.

(b) $\alpha(H) \subset H$. Know: $F = \alpha(F) = \alpha(H * K) = \overbrace{\alpha(H)}^{\text{free factor of } F} * \alpha(K)$

$\overset{(a)}{\Longrightarrow}$ $\alpha(H)$ is a free factor of H. So $\underbrace{H}_{\text{free}} = \overbrace{\alpha(H)}^{\text{free}} * \overbrace{K'}^{\text{free}}$.

Now: $rk(\cancel{H}) \underset{\alpha:\, H \to \alpha(H) \text{ is an iso}}{=} rk\, \alpha(H) + rk(K') = rk(\cancel{H}) + rk(K')$

$rk(K') = 0$, i.e., $K' = \{1\}$. So $H = \alpha(H) * 1 = \alpha(H)$.