Lecture 1 - Prewritten definitions and theorems

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1. Graphs and automorphisms of trees

Def. 1.1:

- A <u>graph</u> X is a tuple consisting of a set of <u>vertices</u> $X^0 \neq \emptyset$, a set of <u>edges</u> X^1 and three maps
$$\alpha : X^1 \to X^0 \quad , \quad \omega : X^1 \to X^0 \quad , \quad \bar{\ } : X^1 \to X^1$$
("beginning", "end" and "inverse" of an edge) such that $\forall e \in X^1$
$$\bar{\bar{e}} = e \quad , \quad \bar{e} \neq e \quad \text{and} \quad \alpha(\bar{e}) = \omega(e) .$$
The vertices $\alpha(e)$ and $\omega(e)$ are called the initial and terminal vertices of the edge e.
- A graph is <u>finite</u> if $X^0 \cup X^1$ is finite.

- A (graph) <u>morphism</u> $p : X \to Y$ between graphs X and Y is a map that sends vertices to vertices, edges to edges and satisfies
$$p(\alpha(e)) = \alpha(p(e)) \quad , \quad p(\omega(e)) = \omega(p(e)) ,$$
$$p(\bar{e}) = \overline{p(e)} .$$
It is called an <u>isomorphism</u> if it is bijective (on vertices and egdes). An automorphism is an isomorphism from a graph to itself.
For $x \in X^0, y \in Y^0$, we also write $p : (X,x) \to (Y,y)$ if $p(x) = y$ (and we want to emphasize this).

- The <u>star</u> of a vertex $x \in X^0$ is the set
$$st(x) := \{ e \in X^1 \mid \alpha(e) = x \} .$$
The <u>valence</u> of x is
$$val(x) := | st(x) | .$$

- A graph is <u>oriented</u> if from each pair $\{e, \bar{e}\}$, $e \in X^1$,

- A graph is <u>oriented</u> if from each pair $\{e, \bar{e}\}$, $e \in X^1$, one element is chosen. This edge is called <u>positively oriented</u>. We write X^1_+ for the set of positively oriented edges and $X^1_- = X^1 \setminus X^1_+$ for its complement, the <u>negatively oriented</u> edges.

- A sequence $l = e_1 e_2 \dots e_n$ of edges of a graph X is called a <u>path of length n</u> if

  $\omega(e_i) = \alpha(e_{i+1}) \quad \forall\ 1 \leq i \leq n-1.$

  In this case, l is called a path from $\alpha(e_1)$ to $\omega(e_n)$ ("initial" and "terminal" vertex of l).

  The path l is <u>closed</u> if $\alpha(e_1) = \omega(e_n)$

  We consider any vertex $v \in X^0$ as a path of length 0 from v to itself.

  A path l is called <u>reduced</u> if it has length 0 or if

  $l = e_1 \dots e_n$ with $e_{i+1} \neq \bar{e}_i \quad \forall\ 1 \leq i \leq n-1.$

- A graph X is <u>connected</u> if for all $v, w \in X^0$, there is a path from v to w.

A <u>circuit</u> in X is a subgraph isomorphic to C_n for some $n \in \mathbb{N}$.

A <u>tree</u> is a connected graph that does not contain a circuit

Lemma 1.3: If X is a connected graph and T is a maximal subtree of X (with respect to inclusion), then T contains all vertices of X.

Def 1.4: A reduced path in a tree is called a <u>geodesic</u>.

Lemma 1.5: If X is a tree and X_1, X_2 are disjoint subtrees, then there is a unique geodesic with initial vertex in X_1, terminal vertex in X_2 and all edges outside X_1 and X_2.

Def. 1.6: Let X be a tree and τ an automorphism of X.
- For $v, w \in X^0$, denote by $[v,w]$ the (unique) geodesic from v to w. Its length is denoted by $d(v,w)$.
- τ acts <u>without inversions</u> if for all $e \in X^1$, we have $\tau(e) \neq \bar{e}$.
- For $x \in X^0 \cup X^1$, we also write $x^\tau := \tau(x)$.

The <u>translation length of τ</u> is defined as
$$|\tau| := \min_{v \in X^0} d(v, v^\tau).$$
If $|\tau| = 0$, we define $\mathring{\tau}$ to be the subgraph of X consisting of all $x \in X^0 \cup X^1$ such that $x^\tau = x$.
If $|\tau| > 0$, define $\vec{\tau}$ as the minimal subtree of X that contains $\{x \in X^0 \mid d(x, x^\tau) = |\tau|\}$.

Theorem 1.7: Let X be a tree and τ an automorphism of X.

i) If $|\tau| = 0$, then $\mathring{\tau}$ is a tree. Let $v \in X^0$ and let $w \in (\mathring{\tau})^0$ be a vertex such that $d(v,w)$ is minimal. Then $d(v,w) = d(v^\tau, w)$ and the concatenation of $[v,w]$ and $[w, v^\tau]$ is the geodesic $[v, v^\tau]$ connecting v and v^τ.

ii) If $|\tau| > 0$ and τ acts without inversions, then $\vec{\tau}$ is isomorphic to L_∞ and τ acts on L_∞ by translation of distance $|\tau|$.
Let $v \in X^0$ and let $w \in \vec{\tau}^0$ be a vertex such that $d(v,w)$ is minimal. Then $[v, v^\tau] \cap \vec{\tau} = [w, w^\tau]$ and $d(v, v^\tau) = |\tau| + 2 \cdot d(v,w)$.