## Lecture 12 - Prewritten definitions and theorems

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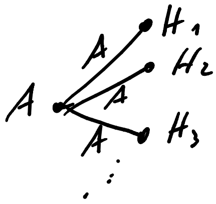
Thm 9.2 (Kurosh): Let $H = \ast_A \{H_i \mid i \in I\}$, i.e. $H = \pi_1(G, Y, Y)$, where $(G, Y)$ is the graph of groups given by




("free product of $\{H_i\}_{i\in I}$ amalgamated over $A$).

Let $G \leq H$ s.t. $G \cap xAx^{-1} = \{1\}$ for all $x \in H$.
Then there exists a free group $F$ and a system of representatives $X_i$ of double cosets $G \backslash H / H_i$ such that

$$G \cong F * \mathop{\ast}_{\substack{i \in I \\ x \in X_i}} (G \cap x H_i x^{-1}).$$

Rem 9.3:

## 10. Applications of Bass-Serre Theory II: Stallings' theorem

References for this section are:
Krön - Cutting up graphs revisited - a short proof of Stallings' structure theorem (Section 10.a)
Dunwoody - Accessibility and groups of cohomological dimension one (Theorem 10.17)
Dunwoody - Cutting up graphs (Section 10.b)

Def 10.1: Let $X$ be a graph.

- Let $E \subseteq X^1$ be a set of edges s.t. if $e \in E$, then $\bar{e} \in E$ as well. We define $X-E$ to be the graph obtained by removing all edges in $E$, i.e. $(X-E)^0 = X^0$, $(X-E)^1 = X^1 \setminus E$.

- A set of vertices $C \subseteq X^0$ is called <u>connected</u> if for all $v, w \in C$, there is a path from $v$ to $w$ in $X$ that has all vertices in $C$.
- A <u>component</u> of $C \subseteq X^0$ is a maximal connected subset.
- $A, B \subseteq X^0$ are <u>separated by</u> $S \subseteq X^0$ if any $v \in A$ and $w \in B$ lie in distinct components of $X^0 \setminus S$.
- $A, B \subseteq X^0$ are <u>separated by</u> $E \subseteq X^1$ if any $v \in A$ and $w \in B$ lie in distinct components of $X-E$.

- A <u>ray</u> is a one-way infinite path $(e_i)_{i \geq 0}$ in $X$ where $e_i \neq e_j$ for $i \neq j$.
- A <u>tail of a ray</u> is an infinite subpath.
- Two rays are said to be <u>separated</u> by a set (of vertices or edges) if the set separates the vertex sets of some tails of the rays.
- Two <u>rays are called equivalent</u> if they cannot be separated by a finite set of edges.
- An <u>end</u> of $X$ is an equivalence class of rays.

Def 10.2: Let $G$ be a finitely generated group.

- For any finite generating set $S$ of $G$, the Cayley graph $Cay(G, S)$ has the same number of ends. This number is called the <u>number of ends of $G$</u>.

Def 10.3: A group $G$ <u>splits over a subgroup</u> $A$ if $G$ is an amalgamated product $G = H *_A K$ or an HNN-extension with associated subgroup $A$.

Aim of this Section.

Thm 10.4 (Stallings): A finitely generated group has more than one end if and only if it splits over some finite subgroup.

## <u>10.a Cuts in graphs</u>

[illegible]

## 10.a Cuts in graphs

Throughout this subsection, fix a connected graph $X$, together with an orientation $X^1_+$.

Def. 10.5:

- For $C, D \subseteq X^0$, let $\delta(C, D)$ denote the set of edges of $X$ that have one endpoint in $C$ and one in $D$.
- For $C \subseteq X^0$, we set $C^c := X^0 \setminus C$ and call $\delta(C) := \delta(C, C^c)$ the <u>edge boundary of</u> $C$.
- A <u>$k$-separator</u> is an edge boundary $\delta(C)$ that contains $k$ positively oriented edges and such that $C$ and $C^c$ are connected.

Lemma 10.6: Let $e \in X^1$ and $k \in \mathbb{N}$.
There are only finitely many $k$-separators that contain $e$.

Def 10.7:

- A <u>cut</u> is a set of vertices $C \subseteq X^0$ such that $\delta C$ is finite and $C$ and $C^c$ are both connected and contains a ray.
- If there is a cut, define
$$\kappa := \min_{C \text{ cut}} \{ |\delta C| \}.$$
- A cut $C$ with $|\delta C| = \kappa$ is called <u>thin</u>.

Lemma 10.8: Let $C$ and $D$ be thin cuts. If $C \cap D$ and $C \cap D^c$ are cuts, then they are thin.