Def. 1.6: Let $X$ be a tree and $\tau$ an automorphism of $X$.

- For $v, w \in X^0$, denote by $[v,w]$ the (unique) geodesic from $v$ to $w$. Its length is denoted by $d(v,w)$.
- $\tau$ acts without inversions if for all $e \in X^1$, we have $\tau(e) \neq \bar{e}$.

The translation length of $\tau$ is defined as

$$|\tau| := \min_{v \in X^0} d(v, \tau(v))$$

If $|\tau| = 0$, we define $\mathring{\tau}$ to be the subgraph of $X$ consisting of all $x \in X^0 \cup X^1$ such that $\tau(x) = x$.
If $|\tau| > 0$, define $\vec{\tau}$ as the minimal subtree of $X$ that contains $\{x \in X^0 \mid d(x, \tau(x)) = |\tau|\}$.

Theorem 1.7: Let $X$ be a tree and $\tau$ an automorphism of $X$.

i) If $|\tau| = 0$, then $\mathring{\tau}$ is a tree. Let $v \in X^0$ and let $w \in (\mathring{\tau})^0$ be a vertex such that $d(v,w)$ is minimal. Then $d(v,w) = d(\tau(v), w)$ and the concatenation of $[v,w]$ and $[w, \tau(v)]$ is the geodesic $[v, \tau(v)]$ connecting $v$ and $\tau(v)$.

ii) If $|\tau| > 0$ and $\tau$ acts without inversions, then $\vec{\tau}$ is isomorphic to $L_\infty$ and $\tau$ acts on $L_\infty$ by translation of distance $|\tau|$.
Let $v \in X^0$ and let $w \in \vec{\tau}^0$ be a vertex such that $d(v,w)$ is minimal. Then $[v, \tau(v)] \cap \vec{\tau} = [w, \tau(w)]$ and $d(v, \tau(v)) = |\tau| + 2 \cdot d(v,w)$.

Def. 1.8: Let $\tau$ be an automorphism of a tree $X$ that acts without inversions. Then $\tau$ is called a *rotation* if $|\tau| = 0$ and a *translation* if $|\tau| > 0$. The subtree $\vec{\tau}$ of a translation is called its *axis*.

Lemma 1.9: Let $X$ be a tree and $T_1, \dots, T_n$ be subtrees of $X$ such that $T_i \cap T_j \neq \emptyset$ $\forall i, j$. Then $\bigcap_{i=1}^{n} T_i \neq \emptyset$.

Prop. 1.10: Let $\tau_1, \dots, \tau_n$ be a finite set of automorphisms of a tree $X$. If $\tau_i$ and $\tau_j \tau_i$ are rotations for all $i, j$, then $\bigcap^{n} \overset{\circ}{\tau}_i \neq \emptyset$.

phisms of a tree $X$. If $\tau_i$ and $\tau_j\tau_i$ are rotations for all $i,j$, then $\bigcap_{i=1}^{n} \mathring{\tau}_i \neq \emptyset$.

Cor. 1.11.: Let $G \leq \mathrm{Aut}(X)$ be a finite group of automorphisms of a tree $X$ that act without inversions. Then <u>$G$ has a global fixed point</u>, i.e. there is $v \in X^0$ s.t. $g(v) = v \quad \forall g \in G$.

## 2. <u>Letting groups act on graphs</u>

Def. 2.1: A <u>group $G$ acts on a graph $X$</u> if $G$ acts on $X^0$ and on $X^1$ such that for all $g \in G$, $e \in X^1$:

Def. 2.1: A group $G$ acts on a graph $X$ if $G$ acts on $X^0$ and on $X^1$ such that for all $g \in G$, $e \in X^1$:

$$g(\alpha(e)) = \alpha(g(e)) \quad \text{and} \quad g(\bar{e}) = \overline{g(e)}.$$

- $G$ acts on $X$ <u>without inversions</u> if $g(e) \neq \overline{g(e)}$ for all $e \in X^1$.

Def. 2.2: For a graph $X$, the <u>barycentric subdivision</u> $B(X)$ is the graph defined as follows:
Replace every edge $e \in X^1$ by two edges $e_1, e_2$ and a new vertex $v_e$ such that

$$\alpha(e_1) = \alpha(e), \quad \omega(e_1) = v_e = \alpha(e_2), \quad \omega(e_2) = \omega(e),$$
$$(\bar{e})_2 = \overline{e_1}, \quad (\bar{e})_1 = \overline{e_2} \quad \text{and } v_{\bar{e}} = v_e.$$

If $G$ acts on $X$, then it also acts on $B(X)$ by setting

$$g(e_1) = (g(e))_1, \quad g(e_2) = (g(e))_2, \quad g(v_e) = v_{g(e)}$$

and preserving the action on the remaining vertices $X^0 \subset B(X)^0$.

Lemma 2.3: $G$ acts on $B(X)$ without inversions.

Def. 2.4: Let $G$ be a group, $S \subseteq G$ a subset.

- We denote by $\Gamma(G, S)$ the oriented graph with vertices and positively oriented edges given by

$$\Gamma(G,S)^0 := G \quad \text{and} \quad \Gamma(G,S)^1_+ := G \times S$$

with

$$\alpha((g,s)) := g, \quad \omega((g,s)) := gs.$$

The negatively oriented edges are given by

$$\Gamma(G,S)^1_- := G \times \{s^{-1} \mid s \in S\}, \quad \overline{(g,s)} := (gs, s^{-1}),$$

where we consider $s^{-1}$ as a formal symbol (in part., $(gs, s^{-1}) \notin G \times S$)

where we consider $s^{-1}$ as a formal symbol (in part., $(g s, s^{-1}) \notin G \times S$).

- If $G = \langle S \rangle$, then $\Gamma(G, S)$ is called the Cayley-graph of $G$ with respect to $S$.

- There is a natural action of $G$ on $\Gamma(G, S)$ given by $g \cdot g' = gg'$ and $g \cdot (g', s) = (gg', s)$ for $g \in G$, $g' \in \Gamma(G, X)^0 = G$, $s \in S$.

Lem 2.5: The action of $G$ on $\Gamma(G, S)$ is free and without inversions.

Lem 2.6: The graph $\Gamma(G, S)$ is connected if and only if $G = \langle S \rangle$.

Def 2.7: Let $G$ be a group that acts on a graph $X$ without inversions.

- For $x \in X^0 \cup X^1$, we denote by $\mathcal{O}(x)$ the orbit of $x$ under $G$.
- The quotient (or factor) graph $G \backslash X$ is the graph with vertex and edge sets

$$(G\backslash X)^0 = \{\mathcal{O}(v) \mid v \in X^0\}$$
$$(G\backslash X)^1 = \{\mathcal{O}(e) \mid e \in X^1\}$$

such that

i) $\alpha(\mathcal{O}(e)) = \mathcal{O}(v)$ if $gv = \alpha(e)$ for some $g \in G$

ii) $\overline{\mathcal{O}(e)} = \mathcal{O}(\bar{e})$

- The map $p: X \to G\backslash X$ is called the projection

ii) $\mathcal{O}(e) = \mathcal{O}(\bar{e})$

- The map $p: X \to G\backslash X$, $x \mapsto \mathcal{O}(x)$ is called the <u>projection</u> and is a graph morphism.
- If $y \in (G\backslash X)^0 \cup (G\backslash X)^1$ and $x \in X^0 \cup X^1$ such that $p(x) = y$, then $x$ is called a <u>lift</u> of $y$.

Proposition 2.8: Let $G$ be a group that acts on a connected graph $X$ without inversions. For every subtree $T' \subseteq G\backslash X =: X'$ of the factor graph, there exists a subtree $T$ in $X$ such that

$$p|_T : T \to T'$$

is an isomorphism.