Lecture 3 - Prewritten definitions and theorems

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3. Elementary properties of free groups and presentations

For a set X, we write $F(X)$ for the free group on X.

Thm 3.1: If $F(X) \cong F(Y)$, then $|X| = |Y|$. We call this cardinal the <u>rank of $F(X)$</u>.

Kor 3.2: If $\varphi : F(Y) \to F(X)$ is surjective, then $|Y| \geq |X|$.

Thm 3.3: Every group is a quotient of a free group.

Def. 3.4: The rank of a group G is defined as
$rk(G) := \min \{ |X| \mid X \subseteq G \, , \langle X \rangle = G \}$
Equivalently
$rk(G) = \min \{ |X| \mid G \cong F(X)/N \text{ for some } N \trianglelefteq F(X) \}$.

Def. 3.5: Let G be a group, $R \subseteq G$. The <u>normal closure</u>
$$R^G := \bigcap_{R \subseteq N \trianglelefteq G} N$$
is the smallest normal subgroup of G that contains R.

We have
$$R^G = \{ \prod_{i=1}^{k} g_i^{-1} r_i^{\varepsilon_i} g_i \mid g_i \in G, r_i \in R, \varepsilon_i \in \{\pm 1\}, k \geq 0 \}.$$

Rem 3.6: If $N \trianglelefteq G$, $x \in N$, then
$uxv \in N \Leftrightarrow uv \in N$.

Def. 3.7: Let G be a group generated by $A = \{a_i \mid i \in I\}$ and let F be the free group on $X = \{x_i \mid i \in I\}$. Then there is an epimorphism and
$$\varphi : F \to G, \quad x_i \mapsto a_i$$
and $G \cong F(X)/N$, where $N = \ker(\varphi)$.

If $R \subseteq N$ s.t. $R^F = N$, then G is uniquely determined by X and R.

if $K = \langle\langle R \rangle\rangle = N$, then it is uniquely determined by X and R.

- We write $G = \langle X \mid R \rangle$ for the <u>presentation</u> of G with <u>generating set X</u> and <u>relations R</u>.
- G is <u>finitely presented</u> if $G = \langle X \mid R \rangle$ for some finite X, R.

Undecidability of the isomorphism and word problem:
There is no algorithm that takes as an input X and R and decides whether $G := \langle X \mid R \rangle$ is isomorphic to the trivial group.
There is no algorithm that decides whether a word w in X represents the identity in G. (This is even true if X and R are finite.)

Thm 3.8: Let G, G' be groups with $G = \langle X \mid R \rangle$.

Thm 3.8: Let G, G' be groups with $G = \langle X \mid R \rangle$. Every map $f: X \to G'$ s.t. $f(r) = 1$ for all $r \in R$ can be extended to a (unique) homomorphism $G \to G'$.

Thm 3.9 (Neumann): If $N \trianglelefteq G$ and N and G/N are finitely presented, then G is finitely presented.