Lecture 4 - Prewritten definitions and theorems

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4. Free groups and graphs

Def 4.1: Let X be a connected graph, $x \in X^0$ and denote by $P(X,x)$ the set of closed paths at x.

- Two paths $p_1, p_2 \in P(X,x)$ are <u>homotopic</u> if p_2 can be obtained from p_1 by a finite number of insertions and deletions of subpaths $e\bar{e}$.
  We write $[p]$ for the <u>homotopy class</u> of p.

- Concatenation of paths defines a multiplication on the set of homotopy classes of paths in $P(X,x)$. The resulting group is called the <u>fundamental group</u> $\pi_1(X,x)$ of X with respect to x.

Rem 4.2:
- This coincides with the fundamental group of the "geometric realisation" of X as defined in topology.
- If $x' \in X^0$, then there is an isomorphism
  $$\pi_1(X,x) \longrightarrow \pi_1(X,x')$$
  $$[p] \longmapsto [q p q^{-1}],$$
  where q is a (fixed) path from x to x'.

- Each homotopy class contains a unique reduced path.

Thm 4.3: Let X be a connected graph, $x \in X^0$ and T a

Thm 4.3: Let X be a connected graph, $x \in X^0$ and T a spanning tree (i.e. a tree containing all vertices). For $v \in X^0$, let p_v be the unique path from x to v in T. Then $\pi_1(X,x)$ is the free group with basis

$$S = \{[p_e] \mid e \in X^1_+ \setminus T\},$$

where $p_e := p_{\alpha(e)}\, e\, p_{\omega(e)}^{-1}$.

Lem. 4.4. Let $G = \langle S \rangle$ be a group. Then the Cayley

Lem. 4.4: Let $G = \langle S \rangle$ be a group. Then the Cayley graph $\Gamma(G, S)$ is a tree if and only if G is a free group with basis S.

Cor 4.5: Every free group acts freely and without inversions on a tree.

Thm 4.6: Let G be a group that acts freely and without inversions on a tree X. Then G is free and its rank is equal to $|Y^1_+ \setminus T|$, where $Y := G \backslash X$ and T is a spanning tree of Y.
In particular, if $Y = G \backslash X$ is finite, then

$$rk(G) = |Y^1_+| - |Y^0| + 1.$$

Cor 4.7 (Nielsen-Schreier):

1) Every subgroup of a free group is free.

2) If G is free of finite rank and H is a subgroup of index m, then

$$rk(H) = m \cdot (rk(G) - 1) + 1.$$