5. Free products and playing Ping-Pong

Def 5.1: Let A, B be groups. Without loss, assume that $A \cap B = \{1\}$.

- A normal form is an expression of the form $g_1 g_2 \cdots g_n$, where $n \geq 0$, $g_i \in (A \cup B) \setminus \{1\}$ and $g_i \in A$ iff $g_{i+1} \in B$. The number n is the length of the normal form and we identify the normal form of length 0 with 1.
- We define a multiplication on the set of normal forms as follows via induction over the length:
  i) $1 \cdot x = x \cdot 1 = x$
  ii) If $x = g_1 \cdots g_n$ and $y = h_1 \cdots h_m$ are normal forms with $n, m \geq 1$, then

$$x \cdot y := \begin{cases} g_1 \cdots g_n h_1 \cdots h_m, & \text{if } g_n \in A, h_1 \in B \text{ or } g_n \in B, h_1 \in A \\ g_1 \cdots g_{n-1} z h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or } g_n, h_1 \in B \text{ and } g_n h_1 = z \neq 1 \\ g_1 \cdots g_{n-1} \cdot h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or } g_n, h_1 \in B \text{ and } g_n h_1 = 1 \end{cases}$$

induction →

- This multiplication makes the set of normal forms into a group, the free product $A * B$.

Rem 5.2: A and B are naturally embedded in $A * B$.

Prop 5.3: If A, B are subgroups of G s.t. every $g \in G$ can be uniquely written as a product $g = g_1 \cdots g_n$, where $\forall i$, $g_i \in (A \cup B) \setminus \{1\}$ and $g_i \in A$ iff $g_{i+1} \in B$, then $G \cong A * B$.

Cor 5.4: If $G = \langle A, B \rangle$ and for every normal form $g = a_1 b_1 \cdots a_k (\cdot b_k)$, one has $g \neq 1$, then $G \cong A * B$.

Thm 5.5 (Torsion & centralisers in free products)

Thm 5.6:
i) If $A = \langle X \mid R \rangle$, $B = \langle Y \mid S \rangle$ with $X \cap Y = \emptyset$, then $A * B \cong \langle X \cup Y \mid R \cup S \rangle$.
ii) The free product is the coproduct in the category of groups. I.e., it can be defined via the following universal property:
Given homomorphisms $\varphi_A : A \to G$ and $\varphi_B : B \to G$, there is a unique hom. $\varphi : A * B \to G$ that makes the following diagram commute:

there is a unique hom. $\varphi: A * B \to G$ that makes the following diagram commute:

$$\begin{tikzcd} A * B \arrow[dr, dashed, "\exists ! \varphi"] & A \arrow[l, hook'] \arrow[d, "\varphi_A"] \\ B \arrow[u, hook] \arrow[r, "\varphi_B"'] & G \end{tikzcd}$$

Thm 5.7: No group can be written both as a non-trivial free product and a non-trivial direct product.

Lemma 5.8 (Ping-Pong): Let G be a group that acts on a set X and let $G_1, G_2 \leq G$ be subgroups with $|G_1| \geq 3$, $|G_2| \geq 2$.

If there are non-empty $X_1, X_2 \subseteq X$, with $X_2 \nsubseteq X_1$ such that

$g(X_2) \subseteq X_1 \quad \forall g \in G_1 \setminus \{1\}$ and

$g(X_1) \subseteq X_2 \quad \forall g \in G_2 \setminus \{1\}$,

then $H := \langle G_1, G_2 \rangle$ is isomorphic to $G_1 * G_2$.

Example: $SL_2(\mathbb{Z})$

Let $SL_2(\mathbb{Z})$ be the group of invertible 2×2-matrices over $\mathbb{Z}$. Its center is

$$Z(SL_2(\mathbb{Z})) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z}) / Z(SL_2(\mathbb{Z}))$$

Thm 5.9: $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

6. Amalgamated products

Def 6.1: Let G, H be groups, $A \leq G$, $B \leq H$ subgroups and $\varphi: A \to B$ an isomorphism. The group
$$G \underset{A=B}{*} H = G \underset{A}{*} H := \langle G * H \mid a = \varphi(a),\ a \in A \rangle$$
is called the amalgamated product of G and H over A.

Def 6.2: Let J_A and J_B be systems coset representatives for A in G and B in H s.t. $1_G \in J_A$, $1_H \in J_B$ (transversals)
An A-normal form in $G \underset{A}{*} H$ (wrt J_A, J_B) is a sequence $(x_0, \dots, x_n)$, $n \geq 0$, where
i) $x_0 \in A$
ii) for $i > 0$, $x_i \in (J_A \setminus \{1_G\}) \cup (J_B \setminus \{1_H\})$ and $x_i \in J_A$ iff $x_{i+1} \in J_B$.

B-normal forms are defined analogously.