

6. Amalgamated products

Def 6.1: Let G, H be groups, $A \leq G$, $B \leq H$ subgroups and $\varphi: A \rightarrow B$ an isomorphism. The group

$$G \underset{A=B}{*} H = G \underset{A}{*} H := \langle G * H \mid \alpha = \varphi(a), a \in A \rangle$$

is called the amalgamated product of G and H over A .

Def 6.2: Let \mathcal{J}_A and \mathcal{J}_B be systems of coset representatives for A in G and B in H p.t. $\Lambda_G \in \mathcal{J}_A$, $\Lambda_H \in \mathcal{J}_B$ (transversals)

An A -normal form in $G \underset{A}{*} H$ (wrt $\mathcal{J}_A, \mathcal{J}_B$) is a sequence

(x_0, \dots, x_n) , $n \geq 0$, where

i) $x_0 \in A$

ii) for $i > 0$, $x_i \in (\mathcal{J}_A \setminus \{\Lambda_G\}) \cup (\mathcal{J}_B \setminus \{\Lambda_H\})$ and $x_i \in \mathcal{J}_A$ iff $x_{i+n} \in \mathcal{J}_B$.

B -normal forms are defined analogously.

Thm 6.3:

Thm 6.4: Any element $f \in F := G *_A H$ can uniquely be written in normal form. I.e. for transversal J_A, J_B , there is a unique A -normal form $(x_0, \dots, x_n) \rightarrow_A f = x_0 \dots x_n$.

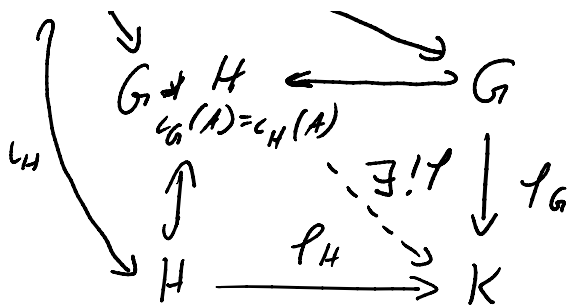
Cor 6.5: Let $F = G \ast_A H$. The canonical projection $G \ast H \rightarrow F$ induces embeddings $G, H \rightarrow F$. The images $c(G), c(H)$ of these embeddings generate F and $c(G) \cap c(H) = c(A) = c(B)$.

Cor. 6.6: If F is a group and $G, H \leq F$, $A \leq G \cap H$ are subgroups such that every $f \in F$ has a unique A -normal form, then $F \cong G \ast_A H$.

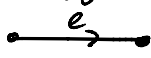
Defn 6.7: The amalgamated product is the cofibre product in the category of groups. I.e., it can be defined via the following universal property:

Let $A \xrightarrow{\iota_G} G$, $A \xrightarrow{\iota_H} H$ being inj. homomorphisms. Then for every pair of homomorphisms $\rho_G: G \rightarrow K$, $\rho_H: H \rightarrow K$ s.t. $\rho_G \circ \iota_G = \rho_H \circ \iota_H$, there is a unique homomorphism $\rho: G \ast_A H \rightarrow K$ such that the following diagram commutes:

$$\begin{array}{ccc}
 A & \xrightarrow{\quad \iota_G \quad} & G \\
 \uparrow & \searrow & \downarrow \\
 \uparrow & G \ast_A H & \xrightarrow{\quad \rho \quad} & G
 \end{array}$$



Def 6.8: A segment is a graph consisting of two vertices and two mutually inverse edges between them.



Lemma 6.9:

Def 6.10 (ad-hoc): Let G be a group, G_1, G_2 subgroups and $A \leq G_1 \cap G_2$. The associated coset graph is the graph defined by

$$X := G/G_1 \cup G/G_2, \quad X^+ := G/A \text{ with}$$

$$\alpha(gA) = gG_1, \quad \omega(gA) = gG_2.$$

Observation 6.11.

1) The edges starting at $g \cdot G_1$ are all of the form $g \cdot g_1 A$,

1) The edges starting at $g \cdot G_1$ are all of the form $g \cdot g_1 A$,
where $g_1 \in \mathcal{JA}$. Analogously for $g \cdot G_2$.

2) There is a 1-to-1 correspondence

$\{ A\text{-normal forms of } g \}$

\cong

$\{ \text{reduced paths in } X \text{ from } 1 \cdot G_1 \text{ to } g \cdot G_1 \}$

Thm 6.12: Let $G = G_1 \times G_2$. Then there is a tree X s.t.
 G acts on X without inversions and such that the
quotient graph $G \backslash X$ is a segment. Moreover, this seg-
ment has a lift to a segment $\overset{\alpha(e)}{\circ} \xrightarrow{e} \overset{w(e)}{\circ}$ in X s.t.
 $G_e = A$, $G_{\alpha(e)} = G_1$ and $G_{w(e)} = G_2$.

Thm 6.13: Let G be a group that acts on a tree X . If the quotient $G \backslash X$ is a segment and let $\tilde{T} = \xrightarrow{\alpha(e)} e \xrightarrow{\omega(e)}$ $\subseteq X$ be a lift of this segment. Then $G \cong G_{\alpha(e)} \ast_{G_e} G_{\omega(e)}$.