Lecture 6 - Prewritten definitions and theorems

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6. Amalgamated products

Def 6.1: Let G, H be groups, $A \leq G$, $B \leq H$ subgroups and $\varphi: A \to B$ an isomorphism. The group

$$G \underset{A=B}{*} H = G \underset{A}{*} H := \langle G * H \mid a = \varphi(a),\ a \in A \rangle$$

is called the <u>amalgamated product</u> of G and H over A.

Def 6.2: Let J_A and J_B be systems of coset representatives for A in G and B in H s.t. $1_G \in J_A$, $1_H \in J_B$ (<u>transversals</u>)

An <u>A-normal form</u> in $G \underset{A}{*} H$ (wrt J_A, J_B) is a sequence $(x_0, \ldots, x_n)$, $n \geq 0$, where

i) $x_0 \in A$

ii) for $i > 0$, $x_i \in (J_A \setminus \{1_G\}) \cup (J_B \setminus \{1_H\})$ and $x_i \in J_A$ iff $x_{i+1} \in J_B$.

B-normal forms are defined analogously.

Rem 6.3:

Thm 6.4: Any element $f \in F := G \ast_A H$ can uniquely be written in normal form. I.e. for transversal T_A, T_B, there is a unique A-normal form $(x_0, \dots, x_n)$ s.t. $f = x_0 \cdot \ldots \cdot x_n$.

Cor 6.5: Let $F = G *_A H$. The canonical projection $G * H \xrightarrow{\iota} F$ induces embeddings $G, H \to F$. The images $\iota(G), \iota(H)$ of these embeddings generate F and $\iota(G) \cap \iota(H) = \iota(A) = \iota(B)$.

Cor. 6.6: If F is a group and $G, H \leq F$, $A \leq G \cap H$ are subgroups such that every $f \in F$ has a unique A-normal form, then $F \cong G *_A H$.

Rem 6.7: The amalgamated product is the cofibre product in the category of groups. I.e., it can be defined via the following universal property:
Let $A \overset{\iota_G}{\hookrightarrow} G$, $A \overset{\iota_H}{\hookrightarrow} H$ be inj. homomorphisms. Then for every pair of homomorphisms $\varphi_G : G \to K$, $\varphi_H : H \to K$ s.t. $\varphi_G \circ \iota_G = \varphi_H \circ \iota_H$, there is a unique homomorphism $\varphi : G \underset{\iota_G(A) = \iota_H(A)}{*} H \to K$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, hook, bend left, "\iota_G"] \arrow[d, hook] & & \\ G * H & & G \arrow[ll, hook'] \end{tikzcd}$$

$$\begin{tikzcd}
G \ast_{\iota_G(A)=\iota_H(A)} H \arrow[dr, dashed, "\exists ! \varphi"] & G \arrow[l] \arrow[d, "\varphi_G"] \\
H \arrow[u] \arrow[r, "\varphi_H"] & K
\end{tikzcd}$$

ι_H

Def 6.8: A <u>segment</u> is a graph consisting of two vertices and two mutually inverse edges between them.

$\bullet \xrightarrow{e} \bullet$

Stern 6.9:

Def 6.10 (ad-hoc): Let G be a group, G_1, G_2 subgroups and $A \leq G_1 \cap G_2$. The associated <u>coset graph</u> is the graph defined by

$$X := G/G_1 \cup G/G_2, \quad X_+^1 := G/A \text{ with}$$

$$\alpha(gA) = gG_1, \quad \omega(gA) = gG_2.$$

Observation 6.11:

1) The edges starting at $g \cdot G_1$ are all of the form $g \cdot g_1 A$,

1). The edges starting at $g \cdot G_1$ are all of the form $g \cdot g_1 A$, where $g_1 \in J_A$. Analogously for $g\, G_2$.

2) There is a 1-to-1 correspondence

$$\{A\text{-normal forms of } g\}$$
$$\cong$$
$$\{\text{reduced paths in } X \text{ from } 1 \cdot G_1 \text{ to } g \cdot G_1\}$$

Thm 6.12: Let $G = G_1 *_A G_2$. Then there is a tree X s.t. G acts on X without inversions and such that the quotient graph $G \backslash X$ is a segment. Moreover, this segment has a lift to a segment $\overset{\alpha(e)}{\circ} \xrightarrow{e} \overset{\omega(e)}{\circ}$ in X s.t. $G_e = A$, $G_{\alpha(e)} = G_1$ and $G_{\omega(e)} = G_2$.

Thm 6.13: Let G be a group that acts on a tree X s.t. the quotient $G \backslash X$ is a segment and let $\tilde{T} = \overset{\alpha(e)}{\bullet} \xrightarrow{e} \overset{\omega(e)}{\bullet} \subseteq X$ be a lift of this segment. Then $G \cong G_{\alpha(e)} \underset{G_e}{*} G_{\omega(e)}$.