Lecture 7 - Prewritten definitions and theorems

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6. Amalgamated products (cont.)

Thm 6.12: Let $G = G_1 *_A G_2$. Then there is a tree X s.t. G acts on X without inversions and such that the quotient graph $G \backslash X$ is a segment. Moreover, this segment has a lift to a segment $\alpha(e) \xrightarrow{e} \omega(e)$ in X s.t. $G_e = A$, $G_{\alpha(e)} = G_1$ and $G_{\omega(e)} = G_2$.

Thm 6.13: Let G be a group that acts on a tree X s.t. the quotient $G \backslash X$ is a segment and let $\tilde{T} = \overset{\alpha(e)}{\bullet} \xrightarrow{e} \overset{\omega(e)}{\bullet} \subseteq X$ be a lift of this segment. Then $G \cong G_{\alpha(e)} \ast_{G_e} G_{\omega(e)}$.

7. HNN extensions

Def 7.1: Let G be a group, $A, B \leq G$, $\varphi: A \to B$ an isomorphism and $\langle t \rangle = \mathbb{Z}$ the infinite cyclic group generated by t.

- The HNN extension of G with respect to A, B and φ is the group

$$G^* = G *_\varphi = \langle G * \langle t \rangle \mid t^{-1} a t = \varphi(a), a \in A \rangle$$

- G is called the base, t the stable letter and A and B are the associated subgroups.

Let $\iota : G * \langle t \rangle \to G^*$ be the canonical proj. Any $x \in G^*$ can be written as

$$x = \iota(g_0)\,\iota(t)^{\varepsilon_1}\,\iota(g_1) \dots \iota(t)^{\varepsilon_n}\,\iota(g_n),$$

where $g_i \in G$, $\varepsilon_j \in \{\pm 1\}$. To shorten notation, we write

$$x = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n.$$

Def 7.2: Let J_A, J_B be transversals of $A\backslash G$, $B\backslash G$ s.t. $1 \in J_A, J_B$. For $g \in G$, let $\bar{g} \in J_A$, $\hat{g} \in J_B$ be the unique elements with $Ag = A\bar{g}$, $Bg = B\hat{g}$.

- A <u>normal form</u> for G^* (wrt J_A and J_B) is a sequence $(g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n)$ such that
  - i) $g_0 \in G$;
  - ii) $\varepsilon_i = -1$ iff $g_i \in J_A$ and $\varepsilon_i = +1$ iff $g_i \in J_B$;
  - iii) there is no consecutive subsequence of the form $t^{\varepsilon}, 1, t^{-\varepsilon}$.

Thm 7.3: Let $G^* = \langle G, t \mid t^{-1}at = \varphi(a), a \in A \rangle$ be an HNN-extension with associated subgroups $A \xrightarrow{\varphi} B$. Then

1) Every $x \in G^*$ can be uniquely written in normal form, i.e. for transversal T_A, T_B, there is a unique normal form $(g_0, t^{\varepsilon_1}, \dots, t^{\varepsilon_n}, g_n)$ such that $x = g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n$.

2) The map $\begin{array}{l} G \to G^* \\ g \mapsto g \end{array}$ is a injective (hence $G \hookrightarrow G^*$ canonically).

If $w = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$, where $n \geq 1$ and there are no subwords $t^{-1} g_i t$, $g_i \in A \cup B$, then $w \neq 1$.

Cor. 7.4: Let $G^* = \langle G, t \mid t^{-1} a t = \varphi(a), a \in A \rangle$ be an HNN-extension with associated subgroups $A \xrightarrow{\varphi} B$. Then the canonical projection $c: G * \langle t \rangle \to G^*$ restricts to isomorphisms on G and $\langle t \rangle$. After identifying these groups with their images, A and B are conjugate in G^* under t.

Cor 7.5: Let H be a group with subgroups G and $\langle t \rangle \cong \mathbb{Z}$, let $A, B \leq H$ such that $t^{-1} A t = B$. If every element of H has a unique normal form, then $H \cong G^*$.

Prop 7.6: Let $H, K \leq G$ and $\varphi: H \to K$ an isomorphism. Then the HNN-extension $G^* = \langle G, t \mid t^{-1} h t = \varphi(h) \rangle$ can be seen as a subgroup of an amalgamated product.

Thm 7.7 (HNN '49): Every countable group C can be embedded in a group G that is generated by two elements and such that:

i) G has n-torsion iff C has n-torsion

ii) If C is finitely presented, so is G

Cor. 7.8 (B. Neumann): There are $2^{\aleph_0}$ many non-isomorphic 2-generator groups.

Thm 7.9 (HNN '49): Every countable group can be embedded into a group where all elements of the same order are conjugate.