Lecture 9 - Prewritten definitions and theorems

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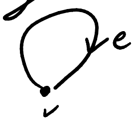
Def 7.12: A loop is a graph consisting of one vertex and two mutually inverse edges:



Rem 7.13:

Thm 7.14: Let $G = H^{*}_{A} = \langle H, t \mid t^{-1} a t = \varphi(a) \ \forall a \in A \rangle$ be an HNN-extension. Then there is a tree X s.t. G acts on X without inversions and such that the quotient graph $G \backslash X$ is a loop. Moreover, there is a segment

$$\tilde{Y} = \underset{\alpha(e)}{\bullet} \xrightarrow{\ e\ } \underset{\omega(e)}{\bullet}$$

in X s.t. $G_e = A$, $G_{\alpha(e)} = H$ and $G_{\omega(e)} = tHt^{-1}$.

Thm 7.15: Let G be a group that acts without inversions on a tree X s.t. the quotient $G\backslash X$ is a loop and let $Y = \bullet \xrightarrow{e} \bullet$ be a lift of $G\backslash X$.
Let $g \in G$ s.t. $g(\alpha(e)) = \omega(e)$. Let $\varphi : G_e \to g^{-1} G_e g$ be the isomorphism given by conjugation with g. Then $g^{-1} G_e g \leq G_{\alpha(e)}$ and the homomorphism

$$\langle G_{\alpha(e)}, t \mid t^{-1} a t = \varphi(a) \ \forall a \in G_e \rangle \to G$$

given by sending t to g is an isomorphism. I.e. G can be written as an HNN-extension $G \cong G_{\alpha(e)}^{*}$.

8. Graphs of groups and general Bass-Serre theory

Def 8.1: A <u>graph of groups</u> (G, Y) consists of a connected graph Y, a <u>vertex group</u> G_v for each vertex $v \in Y^0$, an <u>edge group</u> G_e for each edge $e \in Y^1$, where $G_{\bar{e}} = G_e$, and monomorphisms $\{\alpha_e : G_e \to G_{\alpha(e)} \mid e \in Y^1\}$.
We also write $\omega_e : G_e \to G_{\omega(e)}$ for the monomorphism $\alpha_{\bar{e}} : G_{\bar{e}} = G_e \to G_{\alpha(\bar{e})} = G_{\omega(e)}$.

Def. 8.2: Let (G, Y) be a graph of groups
- We write $F(G, Y)$ for the group defined by
$$\langle G_v, t_e : v \in Y^0, e \in Y^1 \mid t_e t_{\bar{e}} = 1,\ t_e^{-1} \alpha_e(g) t_e = \alpha_{\bar{e}}(g);$$

$$\left\langle G_v, t_e : v \in Y^0, e \in Y^1 \;\middle|\; \begin{array}{l} t_e t_{\bar{e}} = 1,\; t_e^{-1} \alpha_e(g) t_e = \alpha_{\bar{e}}(g); \\ e \in Y^1,\; g \in G_e \end{array} \right\rangle$$

- Let $P \in Y^0$. The <u>fundamental group</u> $\pi_1(G, Y, P)$ <u>of (G, Y) with respect to P</u> is the subgroup of $F(G, Y)$ consisting of all elements of the form

  $g_0 t_{e_1} g_1 t_{e_2} \cdots t_{e_n} g_n$,

  where $e_1 e_2 \ldots e_n$ is a closed path in Y starting at P, $g_0 \in G_P$ and $g_i \in G_{w(e_i)}$.
- Let $T \subseteq Y$ be a spanning tree. The fundamental group $\pi_1(G, Y, T)$ of (G, Y) with respect to T is the quotient of $F(G, Y)$ given by factoring out the normal closure of $\{t_e \mid e \in T^1\}$, i.e.

  $\pi_1(G, Y, T) := \langle F(G, Y) \mid t_e = 1, e \in T^1 \rangle$.

Thm 8.4: Let (G, Y) be a graph of groups, $P \in Y^0$ and $T \subseteq Y$ a spanning tree. The canonical homomorphism $p: F(G, Y) \to \pi_1(G, Y, T)$ restricts to an isomorphism

$T \subseteq Y$ a spanning tree. The canonical monomorphism $p: F(G, Y) \to \pi_1(G, Y, T)$ restricts to an isomorphism $\pi_1(G, Y, P) \to \pi_1(G, Y, T)$.

Cor. 8.5: The fundamental groups $\pi_1(G, Y, P)$ and $\pi_1(G, Y, T)$ are isomorphic for any choice of vertex P and spanning tree T.

Def. 8.6: Let (G, Y) be a graph of groups and $T \subseteq Y$ a spanning tree.

- Let $g \in G_v$ and $g' \in G_u$, where $u, v \in T^0$. We say that <u>g and g' are equivalent</u> (wrt T) if
  $$g' = w_{e_n} \alpha_{\bar{e}_n}^{-1} \cdots w_{e_1} \alpha_{\bar{e}_1}^{-1}(g),$$
  where $e_1 \ldots e_n$ is a path in T from v to u; we also consider g as equivalent to itself.
- Let Y_+^1 be an orientation of Y. Any element $x \in \pi_1(G, Y, T)$ can be written as $x = g_1 \ldots g_n$, where $g_i \in G_v$ for some $v \in Y^0$ or $g_i = t_e^{\pm 1}$ for some $e \in Y_+^1 \setminus T^1$. Such an expression is called <u>reduced</u> if:

$v \in Y^0$ or $g_i = t_e$ for some $e \in Y^1_+$. Such an expression is called <u>reduced</u> if:

i) for no i, g_i and g_{i+1} are equivalent to elements of the same vertex group; (in part. not contained in same v. group)

ii) there is no subword of the form $t_e t_e^{-1}$ or $t_e^{-1} t_e$;

iii) there is no subword of the form $t_e^{-1} g \, t_e$, where g is equivalent to an element from $\alpha_e(G_e)$;

iv) there is no subword of the form $t_e \, g \, t_e^{-1}$, where g is equivalent to an element from $\omega_e(G_e)$.

Rem 8.7:

Thm 8.8: If $g \in \pi_1(G, Y, T)$ has a reduced expression different from 1, then $g \neq 1$. In particular, the vertex groups G_v, $v \in Y^0$ are canonically embedded in $\pi_1(G, Y, T)$.

Def 8.9: Let $p: X \to Y$ be a morphism from a tree X to a connected graph Y and let T be a spanning tree of Y. A pair $(\tilde{T}, \tilde{Y})$ of subtrees in X is called a <u>lift of</u> (T, Y) if $\tilde{T} \subseteq \tilde{Y}$ and

i) each edge from $\tilde{Y}^1 \setminus \tilde{T}^1$ has initial or terminal vertex in $\tilde{T}$;

ii) $p|_{\tilde{T}}: \tilde{T} \to T$ is an isomorphism and p maps $\tilde{Y}^1 \setminus \tilde{T}^1$ bijectively onto $Y^1 \setminus T^1$.

If $v \in Y^0 = T^0$, we write $\tilde{v}$ for its preimage in $\tilde{T}^0$ and for $e \in Y^1$, we write $\tilde{e}$ for its preimage in $\tilde{Y}^1$.

Thm 8.10 (Main theorem of Bass-Serre theory I):
Let (G, Y) be a graph of groups, $T \subseteq Y$ a spanning tree and $G = \pi_1(G, Y, T)$. The group G acts without inversions on on a tree X such that the factor graph $G \setminus X$ is isomorphic to Y and such that the stabilisers of the vertices and edges of X are conjugate to the (canonically embedded) subgroups G_v, $v \in Y^0$, and G_e, $e \in Y^1$.

the vertices and edges of X are conjugate to the (canonically embedded) subgroups G_v, $v \in Y^0$, and G_e, $e \in Y^1$.

Moreover, for the corresponding projection $p : X \to Y = G \backslash X$ there is a lift $(\tilde{T}, \tilde{Y})$ of (T, Y) such that

i) for $\tilde{v} \in \tilde{T}^0$, $\mathrm{Stab}_G(\tilde{v}) = G_v$;

ii) for $\tilde{e} \in \tilde{Y}^1$ with $\alpha(\tilde{e}) \in \tilde{T}^0$, $\mathrm{Stab}_G(\tilde{e}) = \alpha_e(G_e)$;

iii) if $\tilde{e} \in \tilde{Y}^1$ with $\omega(\tilde{e}) \notin \tilde{T}^0$, then $t_e^{-1}(\omega(\tilde{e})) \in \tilde{T}^0$.