Lecture 1 - Script

Dienstag, 23. Februar 2021 07:39

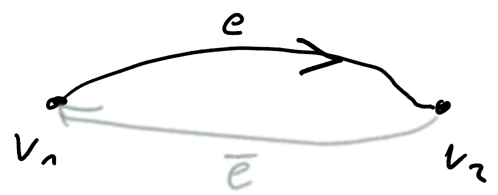
1. Graphs and automorphisms of trees

Def. 1.1:

- A <u>graph</u> X is a tuple consisting of a set of <u>vertices</u> $X^0 \neq \emptyset$, a set of <u>edges</u> X^1 and three maps
  $$\alpha : X^1 \to X^0 \quad , \quad \omega : X^1 \to X^0 \quad , \quad \bar{\ } : X^1 \to X^1$$
  ("beginning", "end" and "inverse" of an edge) such that $\forall e \in X^1$
  $$\bar{\bar{e}} = e \quad , \bar{e} \neq e \quad \text{and} \quad \alpha(\bar{e}) = \omega(e) .$$
  The vertices $\alpha(e)$ and $\omega(e)$ are called the initial and terminal vertices of the edge e.
- A graph is <u>finite</u> if $X^0 \cup X^1$ is finite.

Ex.: $X^0 = \{v_1, v_2\}$, $X^1 = \{e, \bar{e}\}$

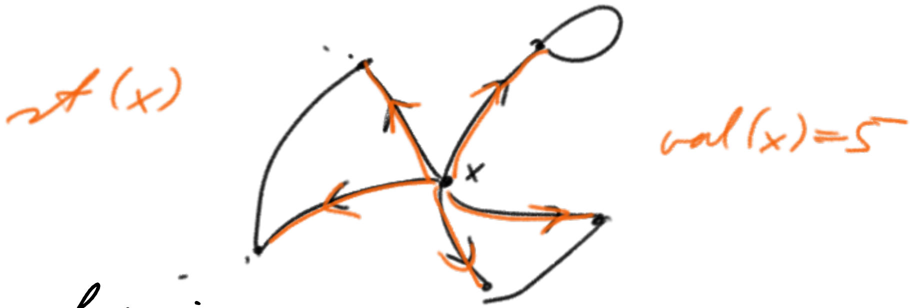
$$\alpha(e) = v_1 \quad , \omega(e) = v_2 \quad , \quad \alpha(\bar{e}) = v \, , \, \omega(\bar{e}) = v_1$$



- A (graph) <u>morphism</u> $p : X \to Y$ between graphs X and Y is a map that sends vertices to vertices, edges to edges and satisfies
  $$p(\alpha(e)) = \alpha(p(e)) \quad , \quad p(\omega(e)) = \omega(p(e)) ,$$
  $$p(\bar{e}) = \overline{p(e)} .$$
  It is called an <u>isomorphism</u> if it is bijective (on vertices and egdes). An automorphism is an isomorphism from a graph to itself.
  For $x \in X^0$, $y \in Y^0$, we also write $p : (X, x) \to (Y, y)$

morphism from a graph is easy.
For $x \in X^0$, $y \in Y^0$, we also write $p : (X,x) \to (Y,y)$ if $p(x) = y$ (and we want to emphasize this).

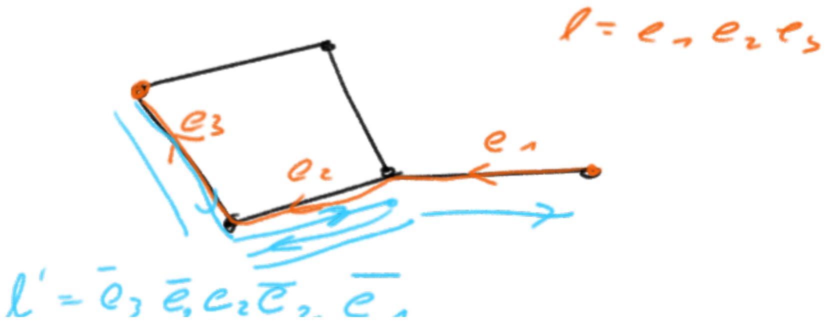
- The <u>star</u> of a vertex $x \in X^0$ is the set
  $st(x) := \{ e \in X^1 \mid \alpha(e) = x \}$.



The <u>valence</u> of x is
$val(x) := | st(x) |$.

- A graph is <u>oriented</u> if from each pair $\{e, \bar{e}\}$, $e \in X^1$, one element is chosen. This edge is called <u>positively oriented</u>. We write X^1_+ for the set of positively oriented edges and $X^1_- = X^1 \setminus X^1_+$ for its complement, the <u>negatively oriented</u> edges.

- A sequence $l = e_1 e_2 \dots e_n$ of edges of a graph X is called a <u>path of length n</u> if
  $\omega(e_i) = \alpha(e_{i+1}) \quad \forall\, 1 \le i \le n-1$.
  In this case, l is called a path from $\alpha(e_1)$ to $\omega(e_n)$ ("initial" and "terminal" vertex of l).
  The path l is <u>closed</u> if $\alpha(e_1) = \omega(e_n)$



$l' = \bar{e}_3 \bar{e}_2 e_2 \bar{e}_2 \bar{e}_1$

We consider any vertex $v \in X^0$ as a path of length 0 from v to itself.

A path l is called <u>reduced</u> if it has length 0 or if

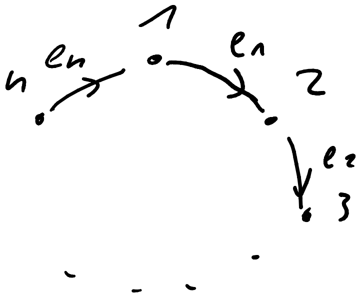
$$l = e_1 \dots e_n \quad \text{with } e_{i+1} \neq \bar{e}_i \quad \forall\, 1 \leq i \leq n-1.$$

- A graph X is <u>connected</u> if for all $v, w \in X^0$, there is a path from v to w.

A <u>circuit</u> /cycle in X is a subgraph isomorphic to C_n for some $n \in \mathbb{N}$.

A <u>tree</u> is a connected graph that does not contain a circuit.

Ex.: <u>C_n</u>:



$X^0 = \{1, \dots, n\}$

$X^1_+ = \{e_1, \dots, e_n\}$

s.t. $\alpha(e_i) = i \quad \forall i$

$\omega(e_i) = i+1 \quad \forall i < n$

$\omega(e_n) = 1$

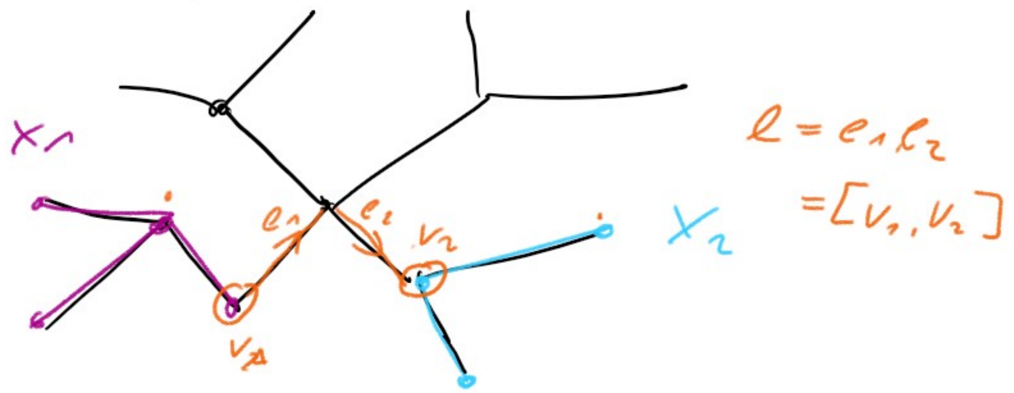
Lemma 1.3: If X is a connected graph and T is a maximal subtree of X (with respect to inclusion), then T contains all vertices of X.

Pf.: Assume that there is a vertex $x \in X^0 \setminus T^0$. As X is connected, there $e \in X^1$ s.t. $\alpha(e) \in T^0$, $\omega(e) \notin T^0$.
But then adding e to T yields a bigger tree ↯. $\square$

Def 1.4: A reduced path in a tree is called a <u>geodesic</u>.

Lemma 1.5: If X is a tree and X_1, X_2 are disjoint subtrees, then there is a unique geodesic with initial vertex in X_1, terminal vertex in X_2 and all edges outside X_1 and X_2.

Pf idea: X



Def. 1.6: Let X be a tree.

- For $v, w \in X^0$, denote by $[v,w]$ the (unique) geodesic from v to w. Its length is denoted by $d(v,w)$.

Definition tbc next lecture.