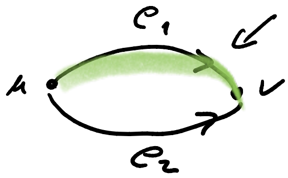
Rem 8.7:

i) Eq. elts. represent the same elt. in $\pi_1(G,Y,T)$.

ii) If $g_1 \dots g_n$ not red., it can be shortened using relations from $\pi_1(G,Y,T)$. In particular, every elt has a reduced sequence.

Ex.: $Y =$ 

$T^1 = \{e_1, \bar{e}_1\}$

$G_u = \langle a \mid a^{12} = 1 \rangle \cong \mathbb{Z}/12\mathbb{Z}$

$G_v = \langle b \mid b^{18} = 1 \rangle$

$G_{e_1} = \langle c \mid c^2 = 1 \rangle$

$G_{e_2} = \langle d \mid d^3 = 1 \rangle$

$\alpha_{e_1} : G_{e_1} \longrightarrow G_u$, $c \longmapsto a^6$

$\omega_{e_1} : G_{e_1} \longrightarrow G_v$, $c \longmapsto b^9$

$\alpha_{e_2} : G_{e_2} \to G_u$, $d \longmapsto a^4$

$\omega_{e_2} : G_{e_2} \to G_v$, $d \longmapsto b^6$

$$\pi_1(G,Y,T) = \left\langle a, b, t \;\middle|\; a^{12} = 1 = b^{18},\ \overset{\alpha_{e_1}(c)}{a^6} = \overset{\omega_{e_1}(c)}{b^9},\ t^{-1} \cdot \underbrace{a^4}_{\alpha_{e_2}(d)} \cdot t = b^6 \right\rangle$$

$(\langle t_{e_2}, t_{\bar{e}_2} \mid t_{e_2} t_{\bar{e}_2} = 1 \rangle = \langle t_{e_2} \rangle)$

$$t b^{12} t^{-1} = a^8$$

$$b t^{-1} a^3 t a^6 b^3 t^{-1} = b t^{-1} a^3 t b^{12} t^{-1} = b t^{-1} a^{11}$$

$$b t^{-1} a^3 t \underbrace{a^6}_{\text{eq. to sth in same vertex gp}} b^3 t^{-1} = b t^{-1} a^3 \underbrace{t\, b^{12}\, t^{-1}}_{\uparrow} = b t^{-1} a^{11}$$

$a^6 = b^9$

$$= \overbrace{b t^{-1} a^{-1}}^{\text{reduced}}$$

$$= b t^{-1} a^{-4} \cdot a^3$$

$$= \underbrace{b^{-5} t^{-1} a^3}_{\text{reduced}}$$

$a^4 t = t b^6$

$\Leftrightarrow t^{-1} a^{-4} = b^{-6} t^{-1}$

$\rightsquigarrow$ The reduced expression here are not necessarily unique.

Thm 8.8: If $g \in \pi_1(G, Y, T)$ has a reduced expression different from 1, then $g \neq 1$. In particular, the vertex groups G_v, $v \in Y^0$ are canonically embedded in $\pi_1(G, Y, T)$.

Pf idea: Use induction on $n = |Y_+^1|$.
If $n = 1$, $\pi_1(G, Y, T)$ is an amalgamated product or an HNN extension and the statement follows Thm 6.4 or Thm 7.3.
For $n > 1$, use Lem. 8.3, iii). $\square$

Def 8.9: Let $p: X \to Y$ be a morphism from a tree X to a connected graph Y and let T be a spanning tree of Y. A pair $(\tilde{T}, \tilde{Y})$ of subtrees in X is called a <u>lift of</u> (T, Y) if $\tilde{T} \subseteq \tilde{Y}$ and
i) each edge from $\tilde{Y}^1 \setminus \tilde{T}^1$ has initial or terminal vertex in $\tilde{T}$;
ii) $p|_{\tilde{T}}: \tilde{T} \to T$ is an isomorphism and p maps $\tilde{Y}^1 \setminus \tilde{T}^1$ bijectively onto $Y^1 \setminus T^1$

ii) $p|_{\tilde{T}}: \tilde{T} \to T$ is an isomorphism and p maps $\tilde{Y}^1 \setminus \tilde{T}^1$ bijectively onto $Y^1 \setminus T^1$.

If $v \in Y^0 = T^0$, we write $\tilde{v}$ for its preimage in $\tilde{T}^0$ and for $e \in Y^1$, we write $\tilde{e}$ for its preimage in $\tilde{Y}^1$.

Thm 8.10 (Structure theorem of Bass-Serre theory I):
Let (G, Y) be a graph of groups, $T \subseteq Y$ a spanning tree and $G = \pi_1(G, Y, T)$. The group G acts without inversions on on a tree X such that the factor graph $G \backslash X$ is isomorphic to Y and such that the stabilisers of the vertices and edges of X are conjugate to the (canonically embedded) subgroups G_v, $v \in Y^0$, and $G_e, e \in Y^1$.

Moreover, for the corresponding projection $p: X \to Y = G \backslash X$ there is a lift $(\tilde{T}, \tilde{Y})$ of (T, Y) such that

i) for $\tilde{v} \in \tilde{T}^0$, $\mathrm{Stab}_G(\tilde{v}) = G_v$;

ii) for $\tilde{e} \in \tilde{Y}^1$ with $\alpha(\tilde{e}) \in \tilde{T}^0$, $\mathrm{Stab}_G(\tilde{e}) = \alpha_e(G_e)$;

iii) if $\tilde{e} \in \tilde{Y}^1$ with $\omega(\tilde{e}) \notin \tilde{T}^0$, then $t_e^{-1}(\omega(\tilde{e})) \in \tilde{T}^0$.

Pf.: We define $X, \tilde{T}, \tilde{Y}$ and the action $G \curvearrowright X$ as follows:
Choose an orientation Y^1_+ of Y. For any vertex $v \in Y^0$, we consider G_v as a subgroup G. Furthermore for every $e \in Y^1_+$, we identify G_e with its image $\alpha(G_e) \leq G_{\alpha(e)} \leq G$.

Let X be the graph defined as follows:

$X^0 := \bigcup G/\ldots \qquad X^1 := \bigcup G/\ldots$

Let X be the graph defined as follows:

$$X^0 := \bigcup_{v \in Y^0} G/G_v \quad , \quad X^1_+ := \bigcup_{e \in Y^1_+} G/G_e$$

$$\alpha(g G_e) := g G_{\alpha(e)} \quad , \quad \omega(g G_e) := g \, t_e \, G_{\omega(e)} \quad \text{for } g \in G, \; e \in Y^1_+.$$

($t_e = 1$ if $e \in T^1$)

$G \curvearrowright X$ by left multiplication.

Note: The valence of $g \cdot G_v$ is

$$\mathrm{val}(g G_v) = \sum_{\substack{e \in Y^1 \\ \alpha(e) = v}} |G_v : \alpha_e(G_e)| .$$

Showing that X is a tree works similarly to Thm 6.12 and Thm 7.14:

X is connected because of existence of reduced forms.

To see that X does not contain a circuit, one checks that a circuit gives a way of writing 1 in red. form. Now use Thm 8.8.

The lift $\tilde{T}$ of T is defined via

$$\tilde{T}^0 = \bigcup_{v \in T^0} \{G_v\} \quad , \quad \tilde{T}^1_+ := \bigcup_{e \in T^1_+} \{G_e\}$$

and

$$\tilde{Y}^0 := \tilde{T}^0 \cup \{ t_e G_{\omega(e)} \mid e \in Y^1_+ \setminus T^1_+ \}$$

$$\tilde{Y}^1_+ := \tilde{T}^1_+ \cup \{ G_e \mid e \in Y^1_+ \setminus T^1_+ \}$$

$\square$

We talked about the following example: (For details, see also the padlet
https://padlet.com/benjaminbrueckmaths/lecture10

Benjamin Brück 2h

A graph of groups

Let

$$\begin{aligned} A &= \langle a \mid a^4 = 1 \rangle \cong \mathbb{Z}/4\mathbb{Z}, \\ B &= \langle b \mid b^6 = 1 \rangle \cong \mathbb{Z}/6\mathbb{Z}, \\ C &= \langle c \mid c^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}, \\ \alpha_C &: C \to A; c \mapsto a^2, \\ \omega_C &: C \to B; c \mapsto b^3. \end{aligned}$$

and use these to define a graph of groups (G,Y) as depicted on the right.

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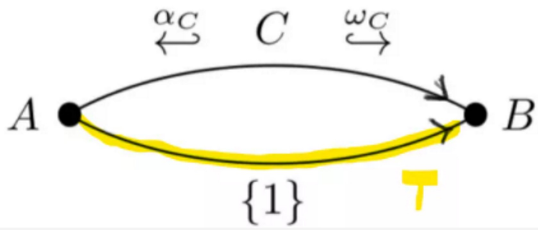
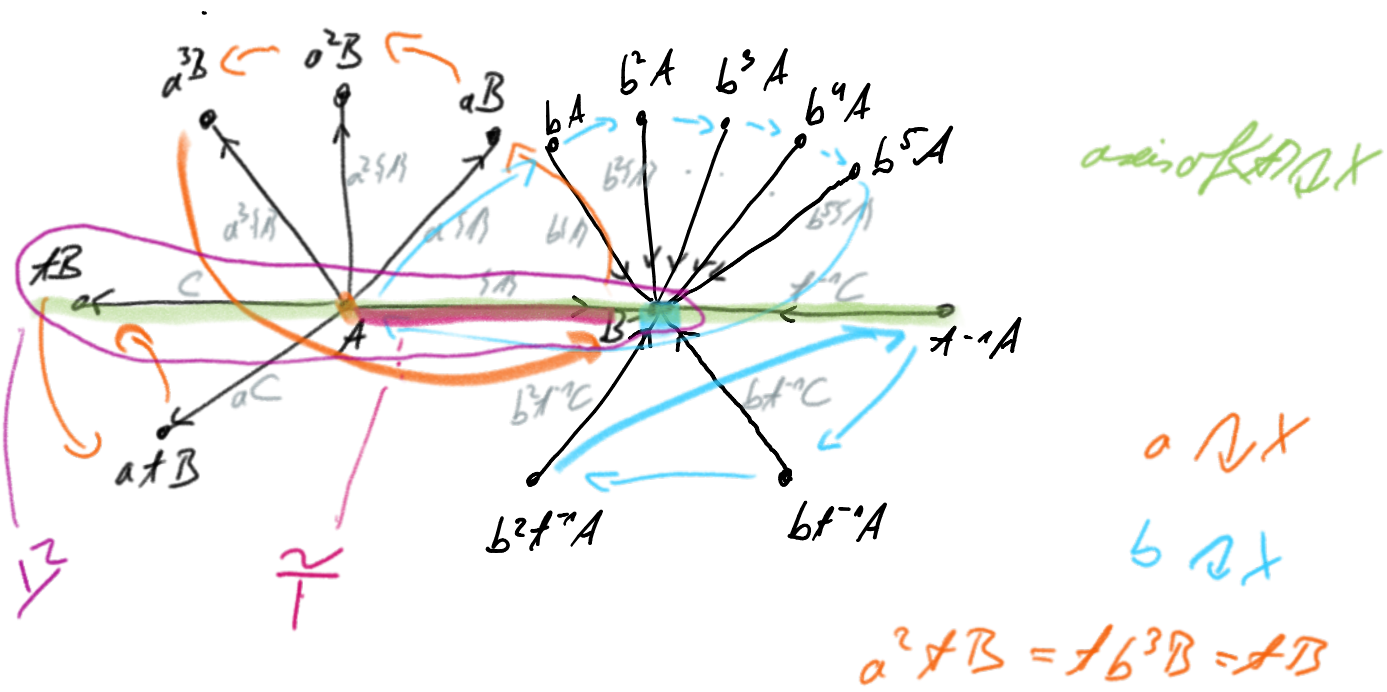
The graph of groups (G,Y)

The yellow edge is a spanning tree T.
(The maps from {1} to A and B are (and must be) given by sending 1 to 1_A and 1_B.)



$$\pi_1(\mathbb{G}, Y, T) = \langle a, b, t \mid a^4 = 1 = b^6,\ t^{-1}a^2t = b^3 \rangle$$

Part of the tree associated to this example:



This shows all edges adjacent to A or B.
For expanding the picture, note that there are only two orbibts of vertices, the one of A and the one of B. This implies that around every vertex, we have a situation isomorphic to the one around A or around B