Cor 8.11: Every finite subgroup of $\pi_1(G,Y,T)$ can be conjugated into a vertex group.

Pf.: By Thm 8.9, $\pi_1(G,Y,T)$ acts on a tree with vertex stabilisers are conj. to the vertex groups of (G,Y). If $H \leq \pi_1(G,Y,T)$, it acts on the same tree. By Cor 1.11, every finite group acting on a tree has a global fixed point, i.e. it is contained in the stabiliser of this vertex. $\square$

Thm 8.12 (Structure theorem II): Let G be a group that acts without inversion on a tree X. Then there is a graph of groups (G,Y) s.t. $G \cong \pi_1(G,Y,T)$ for some (any) spanning tree T of Y.

Pf.: We define (G,Y) as follows:

$X \quad X$

$Y := G\backslash X$, $T \subseteq Y$ spanning tree and $(\tilde{T},\tilde{Y})$ a lift of (T,Y).

$\tilde{T}^0$

For $v \in Y^0$, $G_v := \mathrm{Stab}_G(\tilde{v})$

For $e \in Y^1$, $G_e := \mathrm{Stab}_G(\tilde{e})$

$\tilde{Y}^0$

Def. part of the proof

Embedding α_e:

For each edge $e \in Y^1 \setminus T^1$ with $\omega(\tilde{e}) \notin \tilde{T}^0$, choose $t_e \in G$ s.t. $\omega(\tilde{e}) = t_e \widetilde{\omega(e)}$ and $t_{\bar{e}} = t_e^{-1}$. Now def $e \in Y^1$ the embedding

$$\alpha_e : G_e \longrightarrow G_{\omega(e)}$$

[illegible] $\omega(\tilde{e}) \notin \tilde{T}^0$

$$w_e : G_e \longrightarrow G_{\omega(e)}$$

$$g \longmapsto \begin{cases} g & , \text{if } \omega(\tilde{e}) \in T^0 \\ t_e^{-1} g t_e & , \text{o/w} . \end{cases}$$

The rest of the proof is similar to proof Thm 6.13, and Thm 7.15.

Summary:

1. Show that tree associated to (G, Y) in proof of Thm 8.10 is isomorphic to X.
2. Define homom. $\varphi : \pi_1(G Y, T) \to G$
$$t \longmapsto t$$
$$G_v \longmapsto G_v$$
Use action $G \curvearrowright X$ to show that this is an isom; use normal forms.
3. φ surj.: use existence of normal forms $\hat{=}$ X is connected
4. φ inj.: $\Leftrightarrow \ker \varphi = \{1\} \Leftrightarrow [\varphi(g) = 1 \Rightarrow g = 1]$
$\hat{=}$ uniqueness of normal form of 1.
$\hat{=}$ X has no circuits. $\square$

9. Applications of Bass-Serre Theory I: Subgroups of amalgamated products

Example 9.1: We started working on this example on this week's padlet. In what follows, you find the notes that I took while explaining things in the lecture. More details (and a proper structure for everything) can be found as Example 18.7 in the book of Bogopolski.

Let φ homomorphism from

$$G = \langle a, b \mid a^2 = b^3 \rangle$$

Note that the group G is a special case of the torus knot groups that appeared on Exercise set 3. So it might be helpful to have a look at this again when revising this example.

to Sym(3) -- is given by

$$\varphi(a) = (12)$$

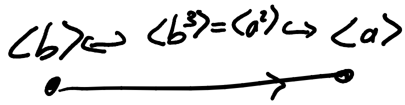
$f(a) = (12)$

$f(b) = (123)$.

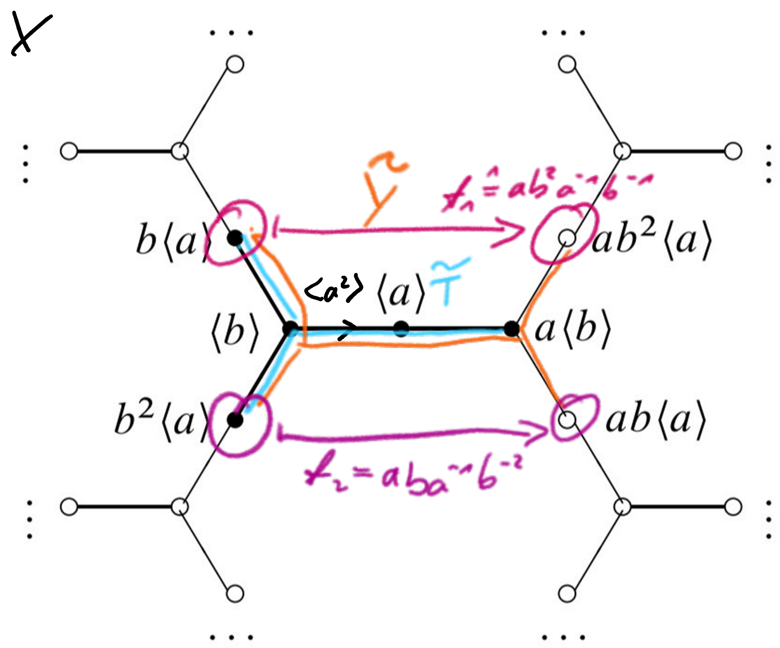
$a^2 \in \ker f$, $b^3 \in \ker f$.

Question: What does $H := \ker f$ look like?

The group G is the fund. group of the gog:



$\rightsquigarrow$ we get an associated Bass-Serre tree (Thm 8.10).



$\rightsquigarrow$ Vertices of X: right cosets of $\langle a \rangle$ and $\langle b \rangle$ in G, edges are given by right cosets of $\langle a^2 \rangle = \langle b^3 \rangle$

H is a subgroup of G $\rightsquigarrow$ $H \curvearrowright X$

What are vertices of $H \backslash X$

System of representatives of left cosets of H in G are given by

$\{1, b, b^2, a, ba, b^2a\}$ (6 elts because

$\{1, b, b^2, a, ba, b^2a\}$ (6 elts because $H\backslash G \cong \mathrm{Sym}(3)$)

$\{1, b, b^2, a, ab, ab^2\}$

Why are these transversals? The following computations are supposed to give you an idea of that:

$\mathrm{Sym}(3) = \langle a, b \mid a^2 = 1 = b^3,\ bab = a \rangle$

$\to a^2 \in H \rightsquigarrow H(1),\ H \cdot (a^2) = H$

$H \cdot b^3 = H$

$H \cdot b,\ Hb^2$

$H \cdot aba \underset{bab=a}{=} H \cdot bab \cdot ba = \underset{a=bab}{Hbab^2a} = Hbbabb^2a$

$= Hb^2ab^3a$
$\underset{b^3=1}{=} Hb^2a^2 \underset{a^2}{=} Hb^2$

vertices in quotient $H\backslash X$: $\underbrace{H \cdot g}\langle a \rangle$ or $\underbrace{H \cdot g}\langle b \rangle$

use list of reprs.

$\rightsquigarrow H \cdot 1 \cdot \langle a \rangle,\ H \cdot b \langle a \rangle,\ H \cdot b^2 \langle a \rangle$

$H \cdot 1 \cdot \langle a \rangle = H \cdot a \langle a \rangle = H \langle a \rangle \qquad H \cdot ba \langle a \rangle = H \cdot b \langle a \rangle$

$\rightsquigarrow$ There are 3 orbits of vertices of X corr. to cosets of $\langle a \rangle$ under the action of H

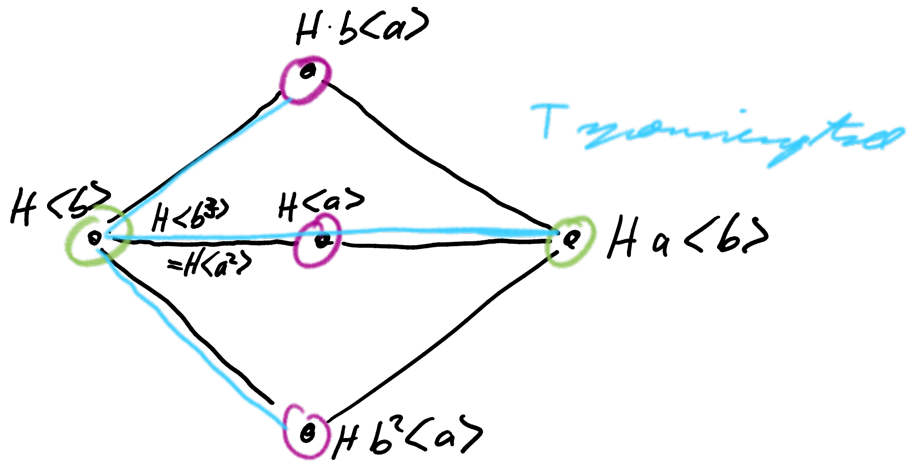
Use other transversal to show:

$\rightsquigarrow$ There are two orbits of X corr. to cosets of $\langle b \rangle$ under the action of H:

$H \cdot 1 \cdot \langle b \rangle,\ H \cdot a \langle b \rangle$ ($H \cdot b \cdot \langle b \rangle = H \langle b \rangle$, $H ab \langle b \rangle = H a \langle b \rangle$)

Now, one needs to check what the edges look like (I did not do this in the lecture; it could be a good exercise for you to do this when going through the example here.)

$\rightsquigarrow H\backslash X$:



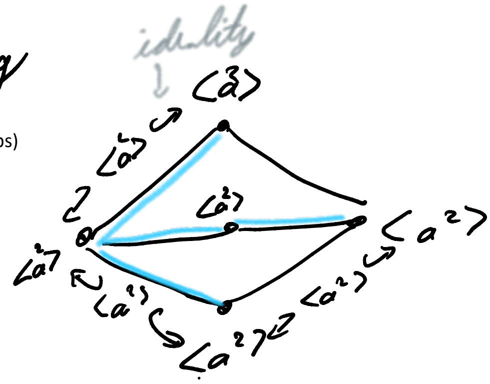
$\rightsquigarrow$ By Thm 8.12, we get a get a graph of groups with underlying graph $H\backslash X$ and vertex & edge groups given by stabilisers of $H \curvearrowright X$.

Now spanning tree $T \subseteq H\backslash X$ and a lift $(\tilde{T}, \tilde{Y})$ in X.

The stabiliser of every edge of $\tilde{Y}$ and every vertex $\tilde{T}$ in H is equal to $\langle a^2\rangle$

$\rightsquigarrow$ get a gog

("gog" = graph of groups)



$$H \cong \pi_1(gog, T) = \left\langle \underset{a^2}{x}, t_1, t_2 \;\middle|\; \begin{matrix} t_1^{-1} x t_1 = x \\ t_2^{-1} x t_2 = x \end{matrix} \right\rangle,$$

where $x \mathrel{\hat{=}} a^2$

$t_1 \mathrel{\hat{=}} ab^2a^{-1}b^{-1}$

$t_2 \mathrel{\hat{=}} aba^{-1}b^{-2}$

$ab^2a^{-1}b^{-1}$ is an element that sends $b\langle a\rangle \in \tilde{T}^0$

$ab^2a^{-1}b^{-1}$ is an element that sends $b\langle a\rangle \in Y^1$ to $ab^2\langle a\rangle \in \tilde{Y}^0$

a sketch about this that shows how to get t1 and t2 can be found in the picture of X above.