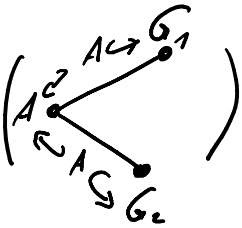
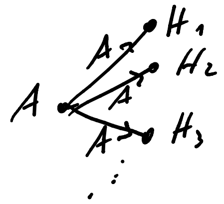
Lecture 12 - Script

Dienstag, 18. Mai 2021 07:48

$$G_1 \ast_A G_2 = \pi_1\left(G_1 \hookleftarrow A \hookrightarrow G_2 \right) = \pi_1 \Bigg($$  $$\Bigg)$$

Thm 9.2 (Kurosh): Let $H = \ast_A \{H_i \mid i \in I\}$, i.e. $H = \pi_1(G, Y, Y)$, where (G, Y) is the graph of groups given by



("free product of $\{H_i\}_{i\in I}$ amalgamated over A")

Let $G \leq H$ s.t. $G \cap xAx^{-1} = \{1\}$ for all $x \in H$.
Then there exists a free group F and a system of representatives X_i of double cosets $G \backslash H / H_i$ such that

$$G \cong F \ast \mathop{\ast}_{\substack{i\in I \\ x \in X_i}} (G \cap x H_i x^{-1}).$$

Note: Kurosh's original thm: $A = \{1\}$

Pf.: Let X be the Bass-Serre tree associated to (G, Y, Y) as in the proof of Thm 8.10, i.e.

$$X^0 = H/A \cup \bigcup_{i\in I} H/H_i \quad , \quad X^1_+ = \bigcup_{i \in I} (H/A \times \{i\}),$$

$$\alpha((hA, i)) = hA \quad , \quad \omega(hA, i) = hH_i .$$

H acts on X by left multipl., hence so does G. This gives rise to a graph of groups with fund. group G as described in the proof of Thm 8.12. It is obtained as follows:

Let $Y := G \backslash X$, $p: X \to Y$ the can. proj., $T \subseteq Y$ a spanning tree and $(\tilde{T}, \tilde{Y})$ a lift of (T, Y) in X.

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Then we have

$$\tilde{T}^\circ = \{xA \mid x \in X_A\} \cup \bigcup_{i \in I} \{xH_i \mid x \in X_i\},$$

where X_A and X_i are systems of representatives of the double cosets $G \backslash H / A$ and $G \backslash H / H_i$.
The stabilisers of these vertices are of the form

$$\mathrm{Stab}_G(xA) = G \cap \underbrace{xAx^{-1}}_{\mathrm{Stab}_H(xA)} = \{1\}$$

$$\mathrm{Stab}_G(xH_i) = G \cap xH_ix^{-1}.$$

The edge stabilisers are trivial as

$$\mathrm{Stab}_G((xA, i)) = G \cap xAx^{-1} = \{1\}.$$

Hence, the resulting gog has trivial edge group and vertex groups given by $G \cap xH_ix^{-1}$, $i \in I$, $x \in X_i$.
Its fundamental group is G, which implies the result. (free group $F \cong \pi_1(Y)$) $\square$

Rem 9.3: There is no analogue of this theorem for direct products. Subgroups of direct prod. are not necessarily direct products themselves.
E.g. $\underbrace{\Delta(G \times G)}_{\cong G} = \{(g, g) \mid g \in G\} \leq G \times G$.

10. Applications of Bass-Serre Theory II: Stallings' theorem

References for this section are:

Krön - Cutting up graphs revisited - a short proof of Stallings' structure theorem (Section 10.a)
Dunwoody - Accessibility and groups of cohomological dimension one (Theorem 10.17)
Dunwoody - Cutting up graphs (Section 10.b)

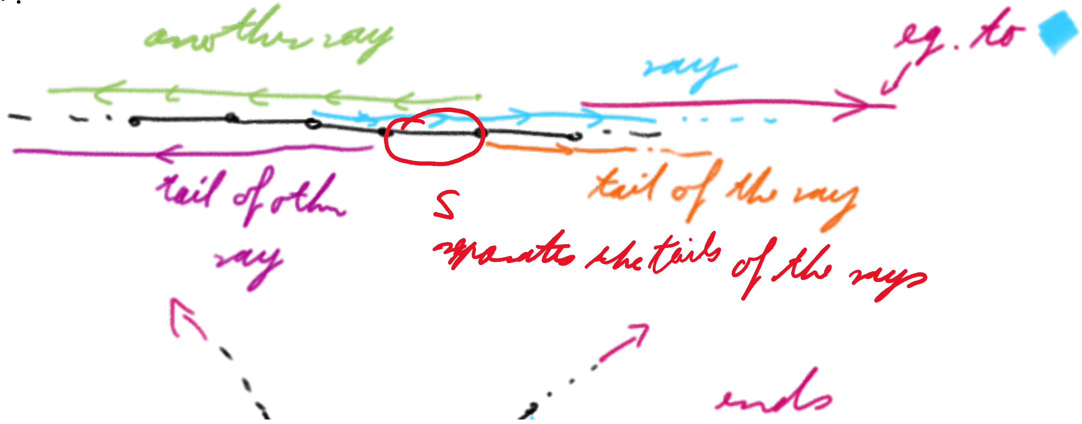
Def 10.1: Let X be a graph.

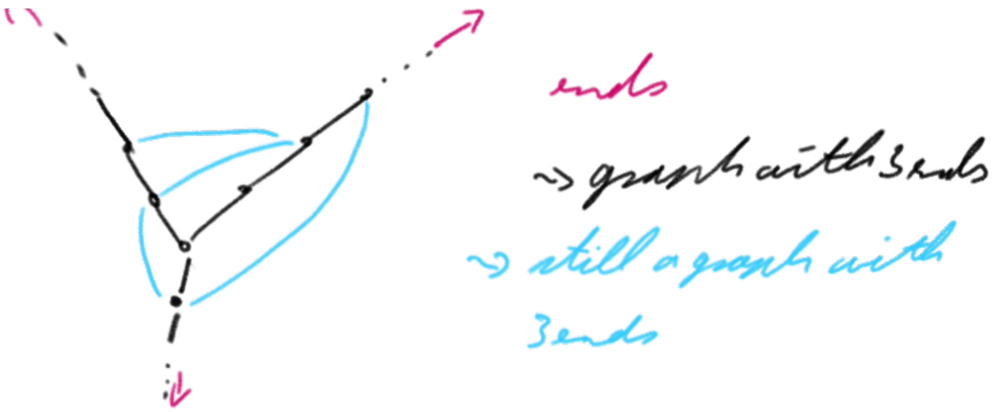
- Let $E \subseteq X^1$ be a set of edges s.t. if $e \in E$, then $\bar{e} \in E$ as well. We define $X - E$ to be the graph obtained by removing all edges in E, i.e. $(X-E)^0 = X^0$, $(X-E)^1 = X^1 \setminus E$.

- A set of vertices $C \subseteq X^0$ is called <u>connected</u> if for all $v, w \in C$, there is a path from v to w in X that has all vertices in C.
- A <u>component</u> of $C \subseteq X^0$ is a maximal connected subset.
- $A, B \subseteq X^0$ are <u>separated by</u> $S \subseteq X^0$ if any $v \in A$ and $w \in B$ lie in distinct components of $X^0 \setminus S$.
- $A, B \subseteq X^0$ are <u>separated by</u> $E \subseteq X^1$ if any $v \in A$ and $w \in B$ lie in distinct components of $X - E$.

- A <u>ray</u> is a one-way infinite path $(e_i)_{i \geq 0}$ in X where $e_i \neq e_j$ for $i \neq j$.
- A <u>tail of a ray</u> is an infinite subpath.
- Two rays are said to be <u>separated</u> by a set (of vertices or edges) if the set separates the vertex sets of some tails of the rays.
- Two <u>rays are called equivalent</u> if they cannot be separated by a finite set of edges.
- An <u>end</u> of X is an equivalence class of rays.

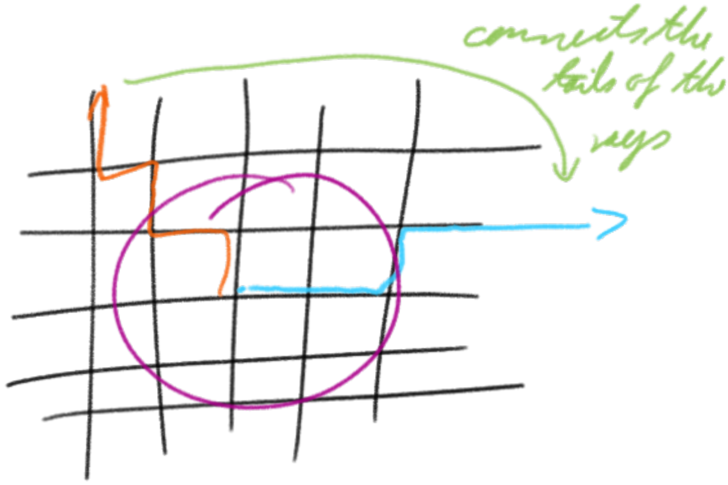
Ex.:



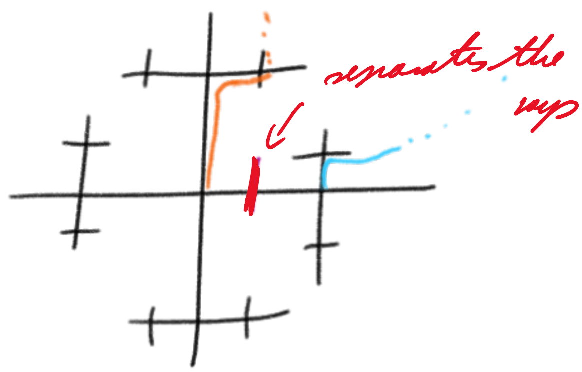


Def 10.2: Let G be a finitely generated group.
- For any finite generating set S of G, the Cayley graph $Cay(G, S)$ has the same number of ends. This number is called the <u>number of ends of G</u>.

Ex.:
- A group has 0 ends iff its finite.
- $\mathbb{Z}$ has 2 ends.
- $\mathbb{Z}^2$ has 1 end.



- F_2 has ∞ many ends.



Fact: A fin. generated groups has either 0, 1, 2 or ∞ many ends.

Def 10.3: A group G <u>splits over a subgroup</u> A if G is an amalgamated product $G = H *_A K$ or an HNN-

Def 10.3: A group G splits over a subgroup A if G is an amalgamated product $G = H *_A K$ or an HNN-extension with associated subgroup A.

Thm 10.4 (Stallings): A finitely generated group has more than one end if and only if it splits over some finite subgroup.

Note: G splits over A iff it acts on a tree with one orbit of pos.-or. edges and edge stabilizer A. (either one or two orbits of vertices)

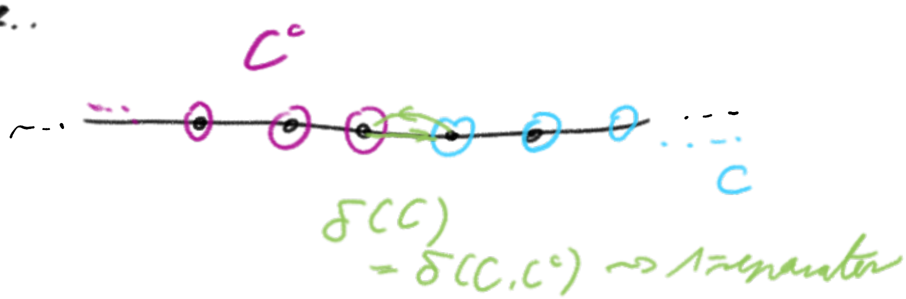
10.a Cuts in graphs

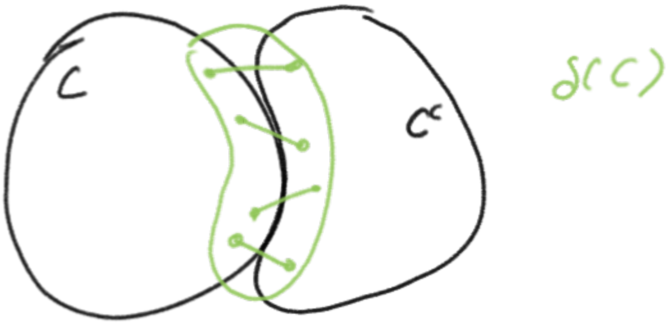
Throughout this subsection, fix a connected graph X, together with an orientation X^1_+.

Def. 10.5:

- For $C, D \subseteq X^0$, let $\delta(C,D)$ denote the set of edges of X that have one endpoint in C and one in D.
$$\delta(C,D) := \{e \in X^1 \mid \alpha(e) \in C \wedge \omega(e) \in D \text{ or } \alpha(e) \in D \wedge \omega(e) \in C\}$$
- For $C \subseteq X^0$, we set $C^c := X^0 \setminus C$ and call $\delta(C) := \delta(C, C^c)$ the edge boundary of C.
- A k-separator is an edge boundary $\delta(C)$ that contains k positively oriented edges and such that C and C^c are connected.

Ex.:





Lemma 10.6: Let $e \in X^1$ and $k \in \mathbb{N}$.
There are only finitely many k-separators that contain e.

Pf.: We prove the statement by induction on k.
$k=1$: trivial ✓

tbc next week