Lemma 10.6: Let $e \in X^1$ and $k \in \mathbb{N}$.
There are only finitely many k-separators that contain e.

Pf.: We prove the statement by induction on k.
$k=1$ is trivial.

$k \to k+1$: Let $\underset{v}{\bullet} \xrightarrow{e} \underset{w}{\bullet}$ be an edge of X.

If there is no $(k+1)$-separator that contains e, there is nothing to prove.

Otherwise, $X - \{e\}$ is connected because $k+1 \geq 2$.

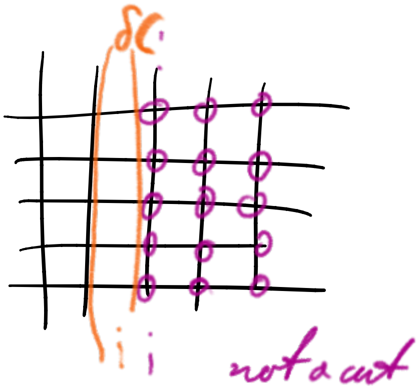
Hence, there is a path p from v to w in $X - \{e\}$. Every $(k+1)$-separator S in X that contains e also contains an edge e' of p. Furthermore, $S \setminus \{e\}$ is a k-separator in $X - \{e\}$. By the induction hypothesis, there are only fin. many k-separators in $X - \{e\}$ that contain e'. The statement follows because p is finite and different $(k+1)$-separators in X that contain e and e' correspond to different k-separators in $X - e$ that contain e'. $\square$

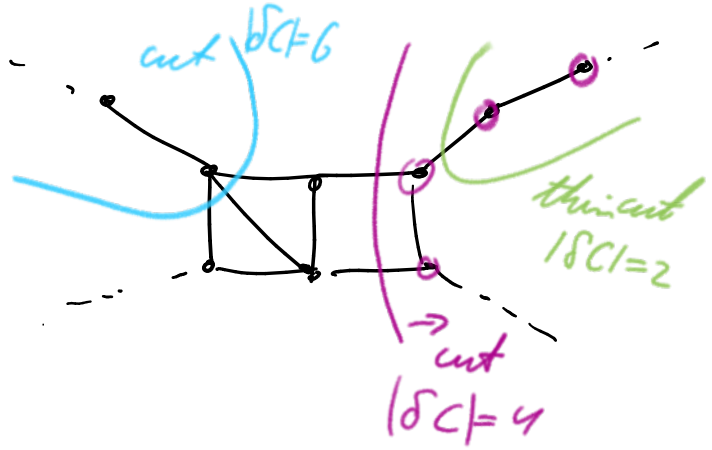
Def 10.7:

- A cut is a set of vertices $C \subseteq X^0$ such that δC is finite and C and C^c both contain a ray.

- If there is a cut, define
  $$\kappa := \min_{C \text{ cut}} \{|\delta C|\}.$$
- A cut C with $|\delta C| = \kappa$ is called <u>thin</u>.

Ex.:



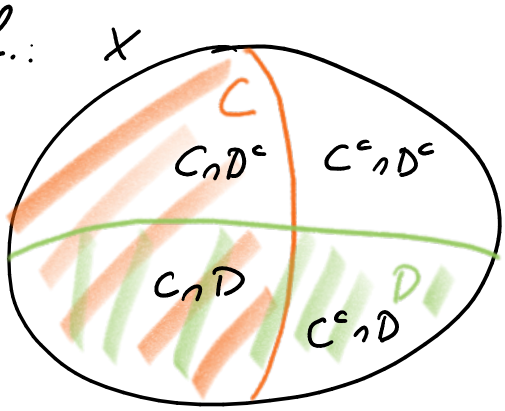


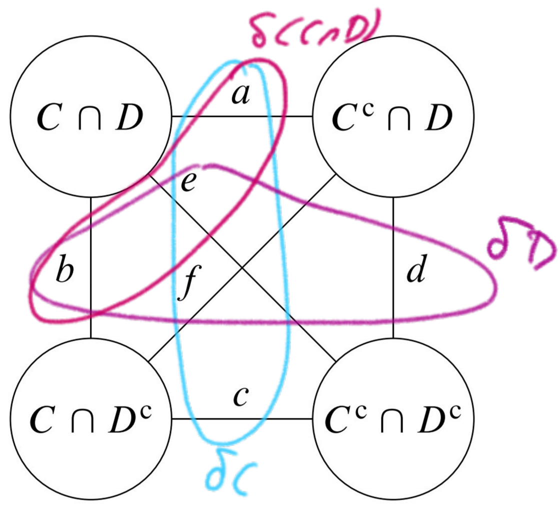
Rem.:

- Cuts are not necessarily connected. Thin cuts are.
- In connected graphs with more than one end, there is always a thin cut.

Lemma 10.8: Let C and D be thin cuts. If $C \cap D$ and $C^c \cap D^c$ are cuts, then they are thin.

Pf.:





Define
$$\begin{aligned}
a &= |\delta(C \cap D, C^c \cap D)|, \\
b &= |\delta(C \cap D, C \cap D^c)|, \\
c &= |\delta(C \cap D^c, C^c \cap D^c)|, \\
d &= |\delta(C^c \cap D, C^c \cap D^c)|,
\end{aligned}$$

Define
$$\begin{aligned}
a &= |\delta(C \cap D, C^c \cap D)|,\\
b &= |\delta(C \cap D, C \cap D^c)|,\\
c &= |\delta(C \cap D^c, C^c \cap D^c)|,\\
d &= |\delta(C^c \cap D, C^c \cap D^c)|,\\
e &= |\delta(C \cap D, C^c \cap D^c)|,\\
f &= |\delta(C \cap D^c, C^c \cap D)|.
\end{aligned}$$

Then
$$\begin{aligned}
\varkappa = |\delta C| &= a+e+f+c\\
= |\delta D| &= b+e+f+d.
\end{aligned}$$

Hence
$$2\varkappa = a+b+c+d+2e+2f. \qquad (*_1)$$

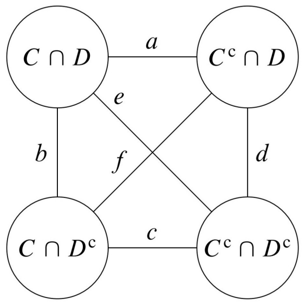
Now if $C \cap D$ and $C^c \cap D^c$ are cuts, then
$$|\delta(C \cap D)| = a+b+e \geq \varkappa \quad \text{and}$$
$$|\delta(C^c \cap D^c)| = c+e+d \geq \varkappa.$$

$\leadsto$ $a+b+c+d+2e \geq 2\varkappa$ and so by $(*_1)$:
$$a+b+c+d+2e = 2\varkappa \text{ and } f=0.$$

Finally, $\underbrace{a+e+b}_{=|\delta(C\cap D)|} = \underbrace{c+e+d}_{=|\delta(C^c \cap D^c)|} = \varkappa$. □

Def 10.9: Let $C, D \subseteq X^\circ$.

- The intersections
$$C \cap D,\ C \cap D^c,\ C^c \cap D,\ C^c \cap D^c$$
are called the <u>corners</u> of C and D. We say that $C \cap D$ is <u>opposite</u> to $C^c \cap D^c$ and $C \cap D^c$ is opposite to $C^c \cap D$.



- C and D are <u>nested</u> if one of their corners is empty.
- <u>C is constant on D</u> if either $D \subseteq C$ or $C \cap D = \emptyset$.

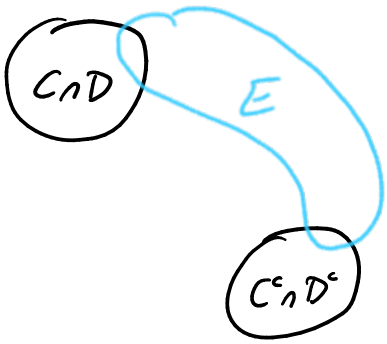
Note: C is nested with D iff C is constant on D or on D^c.

Lem. 10.10: i) If there are two opposite corners of C and D such that E is not nested with either of them, then E is is neither nested with C nor with D.

ii) Let C and D be cuts that are not nested. If a set E is nested with C and D, then E is nested with all corners of C and D.

Pf:

i) If E is not nested with two opposite corners, then it is not constant on these corners. Hence it is not constant



on any of C, C^c, D, D^c.
(E const. on $C \Rightarrow E$ const on any subset of C.)

ii) As C and D are not nested, all their corners are non-empty. After possibly replacing C with C^c and D with D^c, we can assume that E is constant on C and D.
As $C \cap D \neq \emptyset$, this implies that E is const. on

we can assume that E is constant on C and D.
As $C \cap D \neq \emptyset$, this implies that E is const. on

$$C \cup D = (C \cap D) \cup (C^c \cap D) \cup (C \cap D^c).$$

Hence, for any corner A, the set is constant on A or A^c.

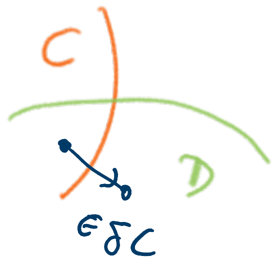
$\square$

Def 10.11: Let C be a cut. We denote by $M(C)$ the set of thin cuts that are not nested with C.
We set $m(C) := |M(C)|$.

Rem.: For all cuts C, $m(C)$ is finite.

I realised after the lecture that my explanations below were probably a bit short. For a more detailed proof that shows that for a thin cut C, the number m(C) is finite (this is all we need), see Dunwoody - "Cutting up graphs" 2.6.

C not nested with $D \Rightarrow D$ contains an edge of δC



By Lemma 10.6, every edge of δC is only contained in fin. many δD, D is a thin cut.

Lem. 10.12: Let C and D be thin cuts that are not nested. If $C \cap D$ and $C^c \cap D^c$ are cuts, then

$$m(C \cap D) + m(C^c \cap D^c) < m(C) + m(D). \quad (*_2)$$

Pf.: By Lem. 10.8, $C \cap D$ and $C^c \cap D^c$ are thin.
Let E be a thin cut. If E is in $M(C \cap D) \cap M(C^c \cap D^c)$, then E is neither nested with $C \cap D$ nor with $C^c \cap D^c$.
Hence by Lem 10.10 i), it is in $M(C)$ and $M(D)$.
This means that if E is counted twice on the LHS of $(*)$, then it is counted twice on the RHS of $(*_2)$.

of $(*)$, then it is counted twice on the RHS of $(*_2)$.

If E is counted once on the LHS of $(*_2)$, then it is in $M(C \cap D) \setminus M(C^c \cap D^c)$ or in $M(C^c \cap D^c) \setminus M(C \cap D)$. Then by Lem. 10.10 ii), it is in $M(C)$ or $M(D)$, and it is counted at least once on the RHS of $(*)$. This shows

$$m(C \cap D) + m(C^c \cap D^c) \leq m(C) + m(D).$$

To see that the inequ. is strict, note that $C \in M(D)$ but $C \notin M(C \cap D)$ or $M(C^c \cap D^c)$, so it's counted on the RHS of $(*_2)$, but not on the LHS. $\square$

Def 10.13: Let $\mathcal{C}$ be the set of all thin cuts and set $m := \min_{C \in \mathcal{C}} \{ m(C) \}$.

A thin cut C with $m(C) = m$ is called <u>optimally nested</u>.

Rem.: As $m(C)$ is finite for all $C \in \mathcal{C}$, the minimum is attained if $\mathcal{C} \neq \emptyset$. I.e. if there is a cut, there is an optimally nested cut.

Thm. 10.14: Optimally nested cuts are all nested with each other.

Pf.: Suppose C and D are optimally nested cuts that are not nested with each other. Then $m \geq 1$

As C and D are cuts, there is a pair of opposite corners that contain a ray and thus are both cuts.

and thus are both cuts.

If $C \cap D$ and $C^c \cap D^c$ are infinite, then they have an infinite connected component and hence contain a ray.
If e.g. $C \cap D$ is finite, then $C \cap D^c$ and $C^c \cap D$ are infinite as C and D are inf.

After possibly replacing C with C^c, we may assume that $C \cap D$ and $C^c \cap D^c$ are cuts. But now Lemma 10.12 says that

$$m(C \cap D) + m(C^c \cap D^c) < m(C) + m(D) = 2m.$$

Hence, one of the summands on the LHS must be smaller than m, which is a contradiction $\square$