Lecture 14 - Script

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10.6 Trees from cuts

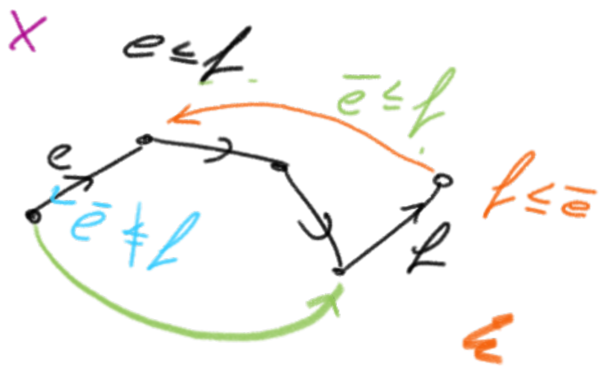
Def 10.15: Let X be a set. A binary relation $\leq$ on X is called a <u>partial order</u> if it satisfies the following:

i) $x \leq x \quad \forall x \in X$ (reflexivity)

ii) $[x \leq y \wedge y \leq x] \Rightarrow x = y$ (antisymmetry)

iii) $[x \leq y \wedge y \leq z] \Rightarrow x \leq z$ (transitivity)

Def 10.16: Let X be a graph. If $e, f \in X^1$, we write $e \leq f$ if there is a reduced path $e = e_1, e_2, \dots, e_n = f$.



Note: the black thing is a tree, the additions in green and orange create circuits in it.

Observation 10.17: If X is a tree, then $\leq$ determines a partial order on X^1. In addition, the following conditions are satisfied:

i) if $e \leq f$, then $\bar{f} \leq \bar{e}$;

ii) if $e \leq f$, there are only finite many $d \in X^1$ for which one has $e \leq d \leq f$;

iii) for any pair e, f, at least one of $e \leq f$, $e \leq \bar{f}$, $\bar{e} \leq f$, $\bar{e} \leq \bar{f}$ holds. → X connected

iv) for no pair e, f, we have $e \leq f$ and $e \leq \bar{f}$; → uniqueness of path

v) for no pair e, f, we have $e \leq f$ and $\bar{e} \leq f$.

→ uniqueness of paths

ii) of partial order uses that

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X is a tree

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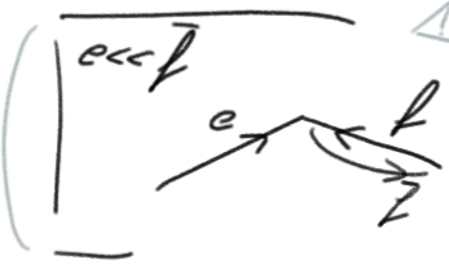
(Dunwoody)
Thm 10.18: Let $(E, \leq)$ be a partially ordered set with a map $E \to E$, $e \mapsto \bar{e}$ s.t. $\bar{\bar{e}} = e$ and suppose that conditions i) – v) from Obs. 10.17 are satisfied.
Then there is a tree T with $E = T^1$ and the order relation on E is precisely the one defined in Def 10.16.

Pf.: We write

- $e < f$ if $e \leq f$ and $e \neq f$
- $e \ll f$ if $e < f$ and if $e \leq d \leq f$, then $e = d$ or $d = f$.

Define a relation $\sim$ on E by
$e \sim f$ if $e = f$ or $e \ll \bar{f}$



⚠ E is not a set!

Claim: $\sim$ is an equivalence relation.
leave out here, but not too hard

Now define T by setting
$T^0 := E/_\sim$, $T^1 := E$
$\omega(e) := [e]$, $\alpha(e) := \omega(\bar{e}) = [\bar{e}]$.

Thus, $\omega(e) = \alpha(f)$ iff $e \ll f$ or $e = \bar{f}$.

By ii), $e \leq f$ iff there is a finite path connecting e and f, so the partial order $\leq$ on E agrees with the one of Def. 10.16. Now iii) implies that T is connected and the antisymmetry of $\leq$ implies that reduced paths are unique. Hence T is indeed a tree. $\square$

Thm 10.4 (Stallings): A finitely generated group G has more than one end if and only if it splits over some finite subgroup.

Pf.: We will show: G acts without inversion on a tree T s.t. there is only one orbit of pos. oriented edges and s.t. every edge stabiliser is finite. (Then use BS theory.)

Let $S \subset G$ be a finite generating set and $X := Cay(G,S)$ the Cayley graph of G wrt S. Because G has more than one end, there is an optimally nested cut C in X.

$$\mathcal{E} := \{gC, gC^c \mid g \in G\} \qquad C \subseteq X^0$$

$(\mathcal{E}, \subseteq)$ is a partially ordered set and $gC \mapsto \overline{gC} := gC^c$ defines a map $\bar{\ }: \mathcal{E} \to \mathcal{E}$ s.t. $\bar{\bar{e}} = e \; \forall e \in \mathcal{E}$.

Claim: $(\mathcal{E}, \subseteq)$ satisfies cond i)-v) of Thm. 10.16.

Pf of claim:

$$gC^c = \{gx \mid x \notin C\} = \{gx \mid gx \notin gC\} = (gC)^c$$

$\rightsquigarrow$ i), iv), v) are obvious

ii) First observe that:

C thin $\Rightarrow$ C^c thin

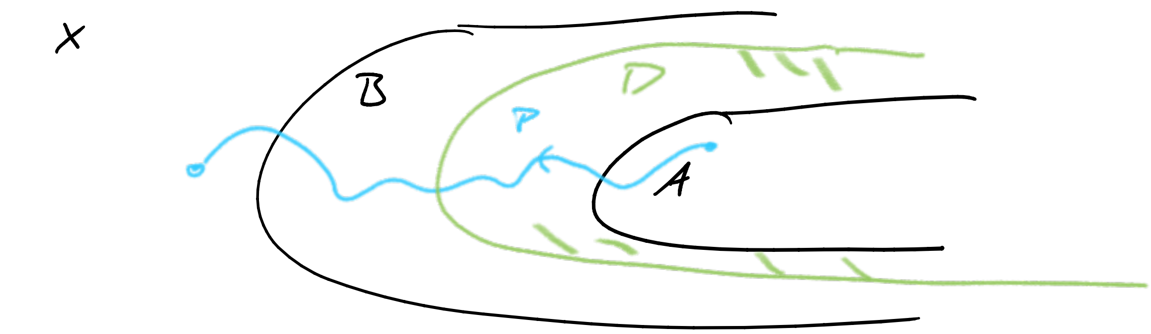
$\Downarrow$

gC thin

$\rightsquigarrow$ $\mathcal{E}$ contains only thin cut

$\rightsquigarrow \mathcal{C}$ contains only this cut

Now let $A, B \in \mathcal{C}$ s.t. $A \subseteq B$.



Let p be a path in X from a vertex in A to a vertex in B^c. Every cut D with $A \subseteq D \subseteq B$ must contain at least one edge of p in its boundary δD. Hence by Lemma 10.6, there are only fin many $D \in \mathcal{C}$ that satisfy this condition

iii) Observe that every $D \in \mathcal{C}$ is optimally nested.

$$m(C) = m(gC) = m(C^c)$$

Now iii) follows from Thm 10.14.

As the claim holds, we can apply Thm 10.17 to get a tree T with edges $T^1 = \mathcal{C}$ and such that the action $G \curvearrowright \mathcal{C}$ extends an action $G \curvearrowright T$. This action has at most 2 orbits of edges.

If $gC \neq C^c$ for all $g \in G$, the action is without inversion and $G \curvearrowright T^1_+$ transitively.

If this is not the case, $G \curvearrowright \mathcal{C}$ transitively. Hence

[illegible]

If this is not the case, $G \curvearrowright C$ transitively. Hence, G acts on the barycentric subdivision $\mathcal{B}(T)$ with at most two orbits of edges.

We now can apply the structure theorem of BS theory (or Thm 6.13 and 7.15) to see that G has the structure of an amalg. prod. or an HNN extension.

Left to show: edge stabilisers of the action $G \curvearrowright T$ are finite. ($\Rightarrow$ stab. on $\mathcal{B}(T)$ are finite as well)

It suffices to show that there are only fin. many $g \in G$ s.t. $gC = C$.

If $gC = C$, then g also stabilises δC. Hence, we have

$$\mathrm{Stab}_G(C) \subseteq \overbrace{\mathrm{Stab}_G}^{H}(\overbrace{\delta C}^{X}).$$

Now δC is a finite set of edges of $X = \mathrm{Cay}(G, S)$. But the action $G \curvearrowright X$ is free, so $\mathrm{Stab}_G(\delta C)$ is finite.

Use: H group, Y set, $H \curvearrowright Y$ freely. Then $H \leq \mathrm{Sym}(Y)$.

$\square$