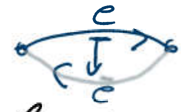
Lecture 2 - Script

Dienstag, 2. März 2021 10:23

Def. 1.6: Let X be a tree and τ an automorphism of X.

- For $v, w \in X^0$, denote by $[v,w]$ the (unique) geodesic from v to w. Its length is denoted by $d(v,w)$.
- τ <u>acts without inversions</u> if for all $e \in X^1$, we have $\tau(e) \neq \bar{e}$.

not: 

The <u>translation length of τ</u> is defined as

$$|\tau| := \min_{v \in X^0} d(v, \tau(v))$$

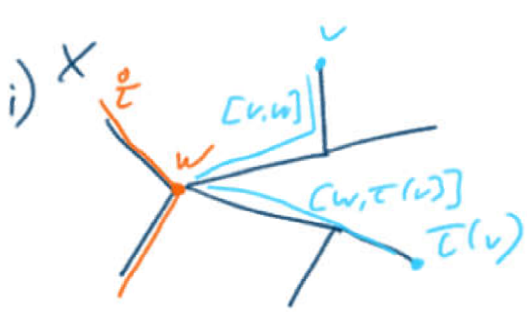
If $|\tau| = 0$, we define $\mathring{\tau}$ to be the subgraph of X consisting of all $x \in X^0 \cup X^1$ such that $\tau(x) = x$.
If $|\tau| > 0$, define $\vec{\tau}$ as the minimal subtree of X that contains $\{x \in X^0 \mid d(x, \tau(x)) = |\tau|\}$.

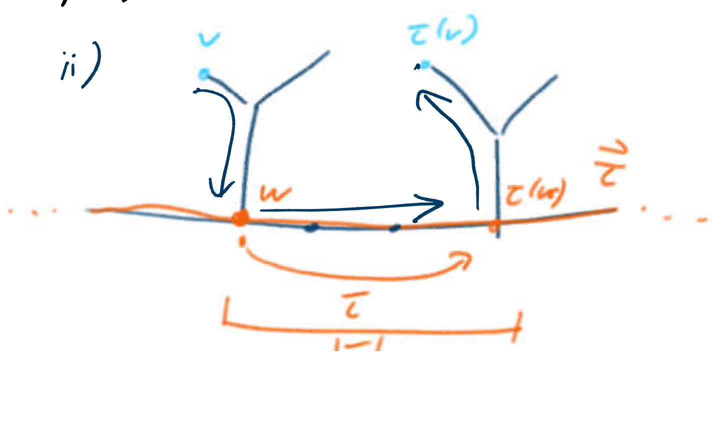
Obs.: $d(\tau(v), \tau(w)) = d(v,w)$

Theorem 1.7: Let X be a tree and τ an automorphism of X.

i) If $|\tau| = 0$, then $\mathring{\tau}$ is a tree. Let $v \in X^0$ and let $w \in (\mathring{\tau})^0$ be a vertex such that $d(v,w)$ is minimal. Then $d(v,w) = d(\tau(v), w)$ and the concatenation of $[v,w]$ and $[w, \tau(v)]$ is the geodesic $[v, \tau(v)]$ connecting v and $\tau(v)$.

ii) If $|\tau| > 0$ and τ acts without inversions, then $\vec{\tau}$ is isomorphic to $\mathcal{L}_\infty$ and τ acts on $\mathcal{L}_\infty$ by translation of distance $|\tau|$.
Let $v \in X^0$ and let $w \in \vec{\tau}^0$ be a vertex such that $d(v,w)$ is minimal. Then $[v, \tau(v)] \cap \vec{\tau} = [w, \tau(w)]$ and $d(v, \tau(v)) = |\tau| + 2 \cdot d(v,w)$.





Pf.: i) If $v, w \in \mathring{\tau}$, then $\tau([v,w])$ is a geodesic from $\tau(v) = v$ and $\tau(w) = w$. As geodesics are unique in trees, $\tau([v,w]) = [v,w]$, so $[v,w] \subset \mathring{\tau}$. Hence, $\mathring{\tau}$ is connected and tree.

The rest follows as for $w \in \mathring{\tau}$

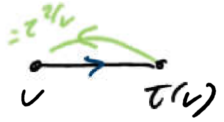
$$d(v,w) = d(\tau(v), \tau(w)) = d(\tau(v), w).$$

ii) Let $v \in X^0$ with $d(v, \tau(v)) = |\tau|$.

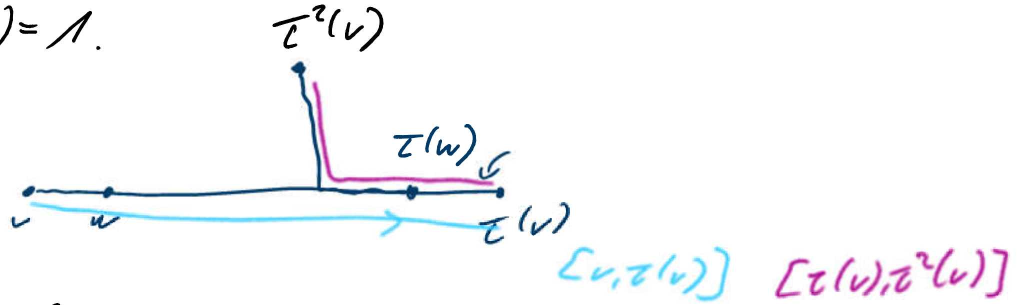
Claim: The last edge of $[v, \tau(v)]$ is not the inverse of the first edge of $[\tau(v), \tau^2(v)]$.

Pf of Claim: Assume it was.

If $|\tau| = 1$, τ would invert the edge $[v, \tau(v)]$. ↯



If $|\tau| > 1$, let w be the vertex on $[v, \tau(v)]$ with $d(v,w) = 1$.



Then $d(w, \tau(w)) = d(v, \tau(v)) - 2 < |\tau|$ ↯

Hence $T = \ldots [\tau^{-1}(v), v][v, \tau(v)][\tau(v), \tau^2(v)] \ldots$ is reduced and isomorphic to C_∞. τ acts on T as translation by $|\tau|$.

If $v \in X^0 \setminus T^0$ and $w \in T$ is of minimal distance to v. Then

v. Then

$$d(v,\tau(v)) = d(v,w) + \underbrace{d(w,\tau(w))} + d(\tau(w),\tau(v))$$
$$= 2d(v,w) + |\tau| > |\tau|.$$

Hence $\vec{\tau} = T$. $\square$

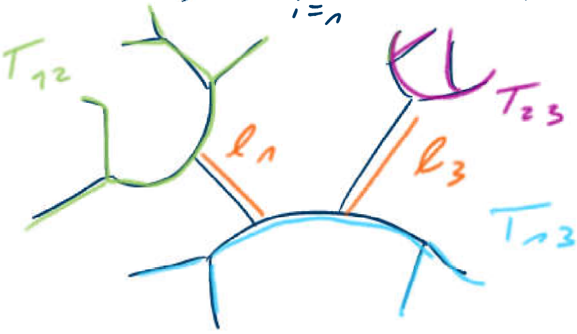
Def. 1.8: Let τ be an automorphism of a tree X that acts without inversions. Then τ is called a <u>rotation</u> if $|\tau| = 0$ and a <u>translation</u> if $|\tau| > 0$. The subtree $\vec{\tau}$ of a translation is called its <u>axis</u>.

Lemma 1.9: Let X be a tree and $T_1, \dots, T_n$ be subtrees of X such that $T_i \cap T_j \neq \emptyset$ $\forall i,j$. Then $\bigcap_{i=1}^{n} T_i \neq \emptyset$.

I skpped the proof of this Lemma in the lecture. Below I include it for completeness, but do not consider it compulsary material that was covered.

Pf. by Induction on n:

<u>$n=3$</u>: Let $T_1, T_2, T_3 \subseteq X$ subtrees and $T_{ij} := T_i \cap T_j \neq \emptyset$.
Then T_{ij} is connected and hence a subtree.
Assume that $\emptyset = \bigcap_{i=1}^{3} T_i = T_{12} \cap T_{13} = T_{12} \cap T_{23} = T_{13} \cap T_{23}$.



Let l_1 be the (unique) geod. connecting T_{12} and T_{13} and l_3 the geod. " T_{13} " T_{23}.
(Lemma 1.5). We have $l_1 \subseteq T_1$, $l_3 \subseteq T_3$.
Then $l_1 \cup T_{13} \cup l_3$ is a tree that contains the geod. l_2 from T_{23} to T_{12}. Hence $l_3 \subseteq l_2 \subseteq T_2$, so $l_3 \subset T_2 \cap T_3$ ↯.

For $n > 3$, apply the same argument to T_1, $\bigcap_{i=2}^{n-1} T_i$ and T $\square$

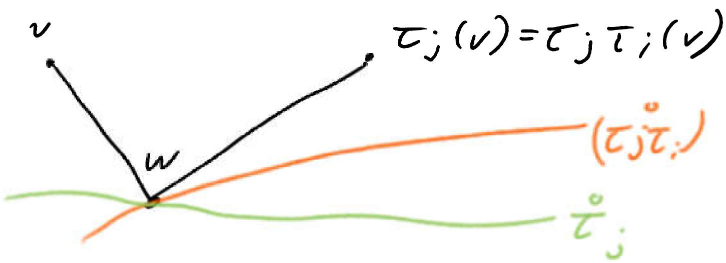
For $n > 3$, apply the same argument to $T_1, \bigcap_{i=2} T_i$, and T_n. $\square$

Prop. 1.10: Let $\tau_1, \dots, \tau_n$ be a finite set of automorphisms of a tree X. If τ_i and $\tau_j \tau_i$ are rotations for all i, j, then $\bigcap_{i=1}^{n} \mathring{\tau}_i \neq \emptyset$.

Pf.: By Lemma 1.9, it suffices to show that $\forall i, j$, $\mathring{\tau}_i \cap \mathring{\tau}_j \neq \emptyset$.

Assume that for i, j, $\mathring{\tau}_i \cap \mathring{\tau}_j = \emptyset$. Let $[v, w]$ be the geod. connecting $\mathring{\tau}_i$ and $\mathring{\tau}_j$. Then

$$[v, \tau_j \tau_i(v)] = [v, \tau_j(v)]$$



By Thm 1.5, the midpoint w of this geodesic is $\mathring{\tau}_j \cap (\tau_j \tau_i)^\circ$, so

$$\tau_j(w) = w = \tau_j \tau_i(w).$$

Hence $w = \tau_i(w)$ and $w \in \mathring{\tau}_i \cap \mathring{\tau}_j$ $\lightning$ $\square$

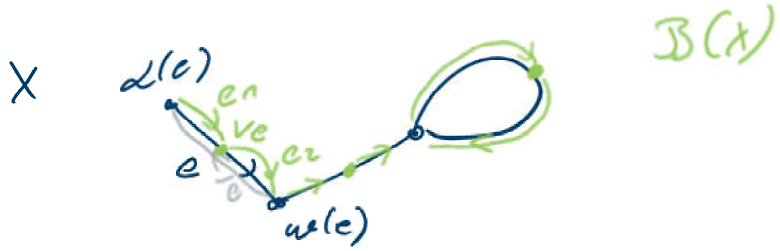
Cor. 1.11: Let $G \leq \mathrm{Aut}(X)$ be a finite group of automorphisms of a tree X that act without inversions. Then <u>G has a global fixed point</u>, i.e. there is $v \in X^0$ s.t. $g(v) = v$ $\forall g \in G$.

Pf.: Check: G contains only rotations as every translation has infinite order ($\tau^n \neq 1$ $\forall n > 0$).

2. Letting groups act on graphs

Def. 2.1: · A group G acts on a graph X if G acts on X^0 and on X^1 such that for all $g \in G$, $e \in X^1$:
$$g(\alpha(e)) = \alpha(g(e)) \quad \text{and} \quad g(\bar{e}) = \overline{g(e)}.$$
· G acts on X without inversions if $g(e) \neq \bar{e}$ for all $e \in X^1$.

Def. 2.2: For a graph X, the barycentric subdivision $\mathcal{B}(X)$ is the graph defined as follows:
Replace every edge $e \in X^1$ by two edges e_1, e_2 and a new vertex v_e such that
$$\alpha(e_1) = \alpha(e), \quad \omega(e_1) = v_e = \alpha(e_2), \quad \omega(e_2) = \omega(e),$$
$$(\bar{e})_2 = \bar{e_1}, \quad (\bar{e})_1 = \bar{e_2} \quad \text{and } v_{\bar{e}} = v_e.$$
If G acts on X, then it also acts on $\mathcal{B}(X)$ by setting
$$g(e_1) = (g(e))_1, \quad g(e_2) = (g(e))_2, \quad g(v_e) = v_{g(e)}$$
and preserving the action on the remaining vertices $X^0 \subset \mathcal{B}(X)^0$.



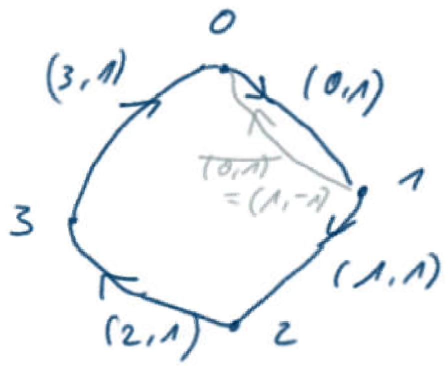
Lemma 2.3: G acts on $\mathcal{B}(X)$ without inversions.
Pf.: Exercise 4.

Def. 2.4: Let G be a group, $S \subseteq G$ a subset.
· We denote by $\Gamma(G, S)$ the oriented graph with vertices and positively oriented edges given by
$$\Gamma(G,S)^0 := G \quad \text{and} \quad \Gamma(G,S)^1_+ := G \times S$$
with
$$\alpha((g,s)) := g, \quad \omega((g,s)) := gs.$$
The negatively oriented edges are given by
$$\Gamma(G,S)^1_- := G \times \{s^{-1} \mid s \in S\}, \quad \overline{(g,s)} := (gs, s^{-1}),$$
where we consider s^{-1} as a formal symbol (in part.,

$\Gamma(G,S)^1_- := G \times \{s^{-1} \mid s \in S\}$, $\overline{(g,s)} := (gs, s^{-1})$,
where we consider s^{-1} as a formal symbol (in part., $(gs, s^{-1}) \notin G \times S$).

- If $G = \langle S \rangle$, then $\Gamma(G,S)$ is called the <u>Cayley-graph of G with respect to S</u>.

$\Gamma(\underbrace{\mathbb{Z}/4\mathbb{Z}}_{G}, \underbrace{\{1\}}_{S})$



pos. edges: (0,1)
(1,1)
(2,1)
(3,1)

- There is a natural action of G on $\Gamma(G,S)$ given by $g \cdot g' = gg'$ and $g \cdot (g', s) = (gg', s)$ for $g \in G$, $g' \in \Gamma(G,X)^0 = G$, $s \in S$.

Rem 2.5: The action of G on $\Gamma(G,S)$ is free and without inversions.

Rem 2.6: The graph $\Gamma(G,S)$ is connected if and only if $G = \langle S \rangle$.

Def 2.7: Let G be a group that acts on a graph X without inversions.

- For $x \in X^0 \cup X^1$, we denote by $\mathcal{O}(x)$ the <u>orbit of x under G</u>.
- The <u>quotient</u> (or <u>factor</u>) <u>graph</u> $G \backslash X$ is the graph with vertex and edge sets
$$(G\backslash X)^0 = \{\mathcal{O}(v) \mid v \in X^0\}$$
$$(G\backslash X)^1 = \{\mathcal{O}(e) \mid e \in X^1\}$$
such that
i) $\alpha(\mathcal{O}(e)) = \mathcal{O}(v)$ if $gv = \alpha(e)$ for some $g \in G$
ii) $\overline{\mathcal{O}(e)} = \mathcal{O}(\bar{e})$
- The map $p: X \to G\backslash X$, $v \mapsto \mathcal{O}(v)$ is called the <u>projection</u>

- The map $p: X \to G\backslash X$, $x \mapsto \mathcal{O}(x)$ is called the <u>projection</u>
and is a graph morphism.
- If $y \in (G\backslash X)^0 \cup (G\backslash X)^1$ and $x \in X^0 \cup X^1$ such that $p(x) = y$, then x is called a <u>lift</u> of y.

Proposition 2.8: Let G be a group that acts on a connected graph X without inversions. For every subtree $T' \subseteq G\backslash X =: X'$ of the factor graph, there exists a subtree T in X such that
$$p|_T : T \to T'$$
is an isomorphism.

$$\begin{tikzcd} X \arrow[d, "p"'] & T \arrow[d, "\cong"] & \text{want} \\ X' = G\backslash X & \supseteq T' & \end{tikzcd}$$

Pf.: Take the set M of all subtrees of X that project injectively into T'. This is a (non-empty) partially ordered set (by inclusion). Every ascending chain in M has a maximal element, given by the union of the trees. By Zorn's lemma, there is a max. elt. T in M. We want to show: $p(T) = T'$.
Assume this was false. Then there is an edge e' with initial point in $p(T)$ and terminal point in $T' \backslash p(T)$.
Let $e \in X^1$ s.t. $p(e) = e'$ and $x := \alpha(e)$. Then by def. $\mathcal{O}(x) = \alpha(e') \in p(T)$. So there is $g \in G$ with $g(x) \in T$. Hence $g(e)$ is an edge with $p(\alpha(g(e))) \in p(T)$ and $p(\omega(g(e))) \notin p(T)$. This implies $T \cup \{g(e)\}$ proj. injectively into T'. ↯ max. $\square$