3. Elementary properties of free groups and presentations

For a set X, we write $F(X)$ for the free group on X.

Thm 3.1: If $F(X) \cong F(Y)$, then $|X| = |Y|$. We call this cardinal the rank of $F(X)$.

Pf.: Let $F(X)^2 = \langle w^2 \mid w \in F(X) \rangle$.

Claim: $F(X)^2 \trianglelefteq F(X)$ normal subgroup

$$F(X)/F(X)^2 \cong H := \left\{ f: X \to \mathbb{Z}/2\mathbb{Z} \;\middle|\; \begin{array}{l} f(x) \neq 0 \text{ for fin.} \\ \text{many } x \in X \end{array} \right\}.$$

Pf of claim: $\quad ((f+g)(x) := f(x) + g(x))$

For $x \in X$ define $f_x \in H$ by

$$f_x(y) := \begin{cases} 1 & , x = y \\ 0 & , \text{o.w.} \end{cases}$$

These f_x generate H. Define

$$\varphi : F(X) \to H$$
$$x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n} \longmapsto \varepsilon_1 f_{x_1} + \ldots + \varepsilon_n f_{x_n}.$$

$x_i \in X,\ \varepsilon_i \in \{\pm 1\}$ $\quad$ (e.g. $X = \{a,b\} \rightsquigarrow a^2 b^{-1} a \rightsquigarrow a^1 a^1 b^{-1} a^1$)

φ is surjective and clearly $F(X)^2 \subseteq \ker \varphi$.
Using the relation

$$xuxv = (xu)^2 u^{-1} v \quad \text{and} \quad x^{-1} u x v = x^{-2} (xu)^2 u^{-1} v,$$

every element in $\ker \varphi$ can be written as a product of squares, so $\ker \varphi = F(X)^2$.

The result now follows as $|H| = \begin{cases} 2^{|X|} & , X \text{ finite} \\ |X| & , \text{o.w.} \end{cases}$ $\square$

$$|H| = \begin{cases} \ldots \\ |X| & \text{, o.w.} \end{cases}$$ □

Kor 3.2: If $\psi : F(Y) \to F(X)$ is surjective, then $|Y| \geq |X|$.

Pf.: $F(Y) \overset{\psi}{\twoheadrightarrow} F(X) \overset{\varphi}{\twoheadrightarrow} F(X)/F(X)^2 = H$ as above.

Then H can be seen as an $\mathbb{F}_2$-vector space, which is generated $\varphi(\psi(Y))$. □

Might expect that: $F(Y) \hookrightarrow F(X)$, then $|Y| \leq |X|$. This is not true (exercise).

Thm 3.3: Every group is a quotient of a free group.

Pf.: Immediate from universal property:

If $G = \langle X \rangle$, there is a unique hom. $F(X) \overset{\varphi}{\to} G$.

φ is surjective, so $G \cong F(X)/\ker\varphi$. □

Def. 3.4: The rank of a group G is defined as

$$rk(G) := \min \{ |X| \mid X \subseteq G, \langle X \rangle = G \}$$

Equivalently

$$rk(G) = \min \{ |X| \mid G \cong F(X)/N \text{ for some } N \trianglelefteq F(X) \}.$$

Def. 3.5: Let G be a group, $R \subseteq G$. The <u>normal closure</u>

$$R^G := \bigcap_{R \subset N \trianglelefteq G} N$$

is the smallest normal subgroup of G that contains R.

We have

$$R^G = \{ \prod_{i=1}^{k} g_i^{-1} r_i^{\varepsilon_i} g_i \mid g_i \in G, r_i \in R, \varepsilon_i \in \{\pm 1\}, k \geq 0 \}.$$

Rem 3.6: If $N \trianglelefteq G$, $x \in N$, then

$uxv \in N \Leftrightarrow uv \in N$.

Def. 3.7: Let G be a group generated by $A = \{a_i \mid i \in I\}$ and let F be the free group on $X = \{x_i \mid i \in I\}$. Then there is an epimorphism and that is an epimorphism and

$$\varphi : F \to G \quad \text{and} \quad G \cong F(X)/N, \text{ where } N = \ker(\varphi).$$
$$x_i \mapsto a_i$$

If $R \subseteq N$ s.t. $R^F = N$, then G is uniquely determined by X and R.

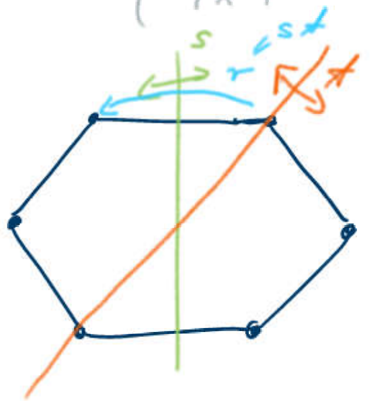
- We write $G = \langle X \mid R \rangle$ for the <u>presentation</u> of G with <u>generating set X</u> and <u>relations R</u>.
- G is <u>finitely presented</u> if $G = \langle X \mid R \rangle$ for some finite X, R.

Ex.: $\mathbb{Z}/n\mathbb{Z} \cong \langle \{x\} \mid \{x^n\} \rangle = \langle x \mid x^n = 1 \rangle$

$$F(\{x\}) \cong \underset{\langle x \rangle}{\mathbb{Z}} \xrightarrow{\varphi} \mathbb{Z}/n\mathbb{Z}$$
$$k \mapsto k + n\mathbb{Z}$$

$$\ker \varphi = \underset{\langle x^n \rangle}{\underset{=}{n\mathbb{Z}}} \subset \mathbb{Z} \qquad \left(\{x^n\}^{\mathbb{Z}} = \langle x^n \rangle \subseteq \mathbb{Z} \right)$$

- Dihedral D_6



$$D_6 = \langle s, t \mid \{s^2, t^2, (st)^6\} \rangle$$
$$= \langle s, r \mid \{s^2, r^6, srsr\} \rangle$$

<u>Undecidability of the isomorphism and word problem:</u>

There is no algorithm that takes as an input X and R and decides whether $G := \langle X \mid R \rangle$ is isomorphic to the trivial group.

There is no algorithm that decides whether a word w in X represents the identity in G (This is even true

There is no algorithm that decides whether a word w in X represents the identity in G. (This is even true if X and R are finite.)

Thm 3.8: Let G, G' be groups with $G = \langle X \mid R \rangle$. Every map $f: X \to G'$ s.t. $f(r) = 1$ for all $r \in R$ can be extended to a (unique) homomorphism $G \to G'$.

Pf.: Any $g \in G$ can be written as $g = \underbrace{x_1 \cdots x_k}_{w}$, where $x_i \in X^{\pm}$.

Define $f'(g) := f(x_1) \cdots f(x_k)$.

To see that f' is well-defined, note that $f'(g) = 1$ for all $g \in R^{F(X)}$.

If $g = \underbrace{x_1' \cdots x_\ell'}_{w'}$, then $(x_1 \cdots x_k)^{-1}(x_1' \cdots x_\ell') \in R^{F(X)}$, so $f(w^{-1} w') = f(w^{-1}) f(w') = 1$. $\square$

Thm 3.9 (Neumann): If $N \trianglelefteq G$ and N and G/N are finitely presented, then G is finitely presented.

Pf.: Let $N = \langle x_1, \ldots, x_m \mid r_1 = \ldots = r_k = 1_G \rangle$

$$G/N = \langle y_1 N, \ldots, y_n N \mid s_1 = \ldots = s_\ell = 1_{G/N} \rangle$$

Interpret $r_i = r_i(\overline{x})$ as words in letter $\overline{x} = (x_1, \ldots, x_m)$

" $s_j = s_j(\overline{yN})$ " " " $\overline{yN} = (y_1 N, \ldots, y_n N)$

Then G is generated by $x_1, \ldots, x_m, y_1, \ldots, y_n$ and for appropriate words t_j, u_{ij}, v_{ij} in $\overline{x}$, the following relations hold in G:

$$R := \begin{cases} r_i(\overline{x}) = 1_G \ , & s_j(\overline{y}) = t_j(\overline{x}) \in N \\ y_j^{-1} x_i y_j = u_{ij}(\overline{x}) \in N \ , & y_j^{-1} x_i^{-1} y_j = v_{ij}(\overline{x}) \in N. \end{cases}$$

Let

$$G' := \langle x_1, \dots, x_m, y_1, \dots, y_n \mid R \rangle .$$

Then by Thm 3.8, there is an epim.

$$\alpha : G' \to G$$
$$x_i \mapsto x_i$$
$$y_j \mapsto y_j$$

Claim: α is an iso., i.e. $K := \ker \alpha$ is trivial.

Let $N' := \langle x_1, \dots, x_m \rangle \leq G'$. Then $\alpha|_{N'} : N' \to N$ is an iso. as $N = \langle x_1, \dots, x_m \mid r_1 = \dots = r_k \rangle$.

Hence, $N' \cap K = \{1_G\}$. As $N' \trianglelefteq G'$, this implies that α induces an epim.

$$\alpha' : G'/N' \to G/N$$
$$y_i N' \mapsto y_i N$$

and $\ker \alpha' \cong K$.

But $G/N = \langle y_1 N, \dots, y_n N \mid s_1 = \dots = s_k \rangle$, so α' is an iso. $\square$