Lecture 4 - Skript

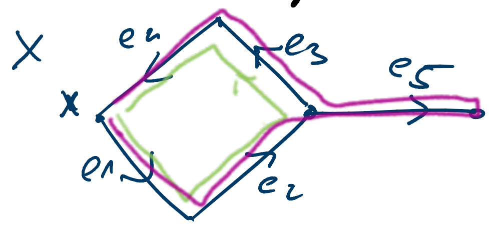
Dienstag, 16. März 2021 07:52

4. Free groups and graphs

Def 4.1: Let X be a connected graph, $x \in X^0$ and denote by $P(X,x)$ the set of closed paths at x.

- Two paths $p_1, p_2 \in P(X,x)$ are <u>homotopic</u> if p_2 can be obtained from p_1 by a finite number of insertions and deletions of subpaths $e\bar{e}$.
  We write $[p]$ for the <u>homotopy class</u> of p.

Ex.:



$[e_1 e_2 e_3 e_4] = [e_1 e_2 e_5 \bar{e}_5 e_3 e_4]$

$\in P(X,x)$

$[e_1 e_2 e_3 e_4]^{-1} = [\bar{e}_4 \bar{e}_3 \bar{e}_2 \bar{e}_1]$

- Concatenation of paths defines a multiplication on the set of homotopy classes of paths in $P(X,x)$. The resulting group is called the <u>fundamental group</u> $\pi_1(X,x)$ of X with respect to x.

Rem 4.2:

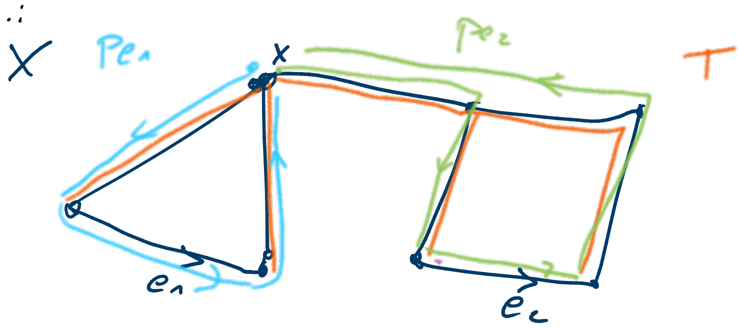
- This coincides with the fundamental group of the "geometric realization" of X as defined in topology.
- If $x' \in X^0$, then there is an isomorphism
$$\pi_1(X,x) \longrightarrow \pi_1(X,x')$$
$$[p] \longmapsto [q p q^{-1}],$$
where q is a (fixed) path from x to x'.

- Each homotopy class contains a unique reduced path.

Thm 4.3: Let X be a connected graph, $x \in X^0$ and T a <u>spanning tree</u> (i.e. a tree containing all vertices). For $v \in X^0$, let p_v be the unique path from x to v in T.

spanning tree (i.e. a tree containing all vertices). For $v \in X^0$, let p_v be the unique path from x to v in T. Then $\pi_1(X,x)$ is the free group with basis
$$S = \{[p_e] \mid e \in X^1_+ \setminus T\},$$
where $p_e := p_{\alpha(e)} e \, p_{\omega(e)}^{-1}$.

Ex.:



Pf.: If $p = e_1 \dots e_n$ is a closed path in X starting at x, then $[p] = [p_{e_1}][p_{e_2}]\dots[p_{e_n}]$. For $e \in T^1$, then $[p_e] = 1$. Hence S generates $\pi_1(X,x)$.

To see that the group is free, we show that every element has a unique reduced form. For this let $[p] = \underset{\in S}{[p_{e_1}]} \cdot \ldots \cdot [p_{e_n}]$ be reduced with respect to S, i.e. $[p_{e_i}]^{-1} \neq [p_{e_{i+1}}]$ $\forall i$. Then $e_{i+1} \neq \bar{e}_i$ $\forall i$. It follows that when reducing $p = p_{e_1} \cdots p_{e_n}$, none of the e_i is cancelled out. Hence, p is homotopic to a reduced path of the form $t_1 e_1 t_2 e_2 \dots e_n t_{n+1}$, where $t_i \subseteq T$. Since every homotopy class contains a unique reduced path, the sequence $e_1, \dots, e_n$ is uniquely determined by $[p]$. So the reduced form $[p] = [p_{e_1}] \cdot \ldots \cdot [p_{e_n}]$ is unique. $\square$

Lem. 4.4: Let $G = \langle S \rangle$ be a group. Then the Cayley graph $\Gamma(G,S)$ is a tree if and only if G is a free group with basis S.

Pf. idea: A conn. graph is a tree iff all red. paths are group with basis S.

Pf idea: A conn. graph is a tree iff all red. paths are unique.

Use: Finding a path in $\Gamma(G,S)$ from id to g is equivalent to writing g as a product of elements in $S^{\pm}$. $\square$

Cor 4.5: Every free group acts freely and without inversions on a tree.

Thm 4.6: Let G be a group that acts freely and without inversions on a tree X. Then G is free and its rank is equal to $|Y_+^1 \setminus T|$, where $Y := G\backslash X$ and T is a spanning tree of Y.

In particular, if $Y = G\backslash X$ is finite, then

$$rk(G) = |Y_+^1| - |Y^0| + 1.$$

Pf.: Let $\tilde{T}$ be a lift of T in X.

$$\begin{array}{ccc} X & \supseteq & \tilde{T} \\ \downarrow p & & \downarrow \\ Y = G\backslash X & \supseteq & T \end{array}$$

Then (by definition) distinct vertices of $\tilde{T}$ lie in distinct G-orbits and every G-orbit has a representative in $\tilde{T}$. Choose an orientation Y_+^1 of Y and let X_+^1 be the induced orientation on X.

Let $E = Y_+^1 \setminus T$. For every $e \in E$, there is a lift $\tilde{e}$ with $\alpha(\tilde{e}) \in \tilde{T}$. This lift is unique:

$\alpha(\tilde{e}) \in \tilde{T}$. This lift is unique:

G acts transitively on the set

$\{e' \in X^\wedge \mid p(e') = e\}$,

i.e. for each such e', there is $g \in G$ with $g\hat{e} = e'$.

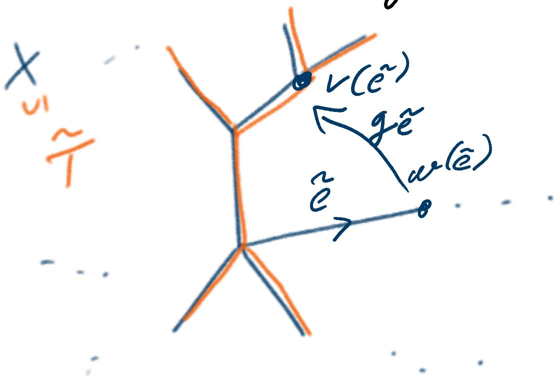
But $\tilde{T}$ contains only one vertex from each G-orbit, so if $\alpha(e') \in \tilde{T}$, we have

$$\alpha(\tilde{e}) = \alpha(e') = \alpha(g\tilde{e}) = g(\alpha(\tilde{e})).$$

As G acts freely, this implies that $g = 1$, so $\hat{e} = e'$.

Let $\tilde{E} := \{\tilde{e} \in X_+^\wedge \mid \alpha(\tilde{e}) \in \tilde{T}, \omega(\tilde{e}) \notin \tilde{T}\}$. Then by the considerations above, the proj. map induces a bijection $p : \tilde{E} \to E$.

For $\tilde{e} \in \tilde{E}$, there is a unique vertex $v(\tilde{e}) \in \tilde{T}$ with $v(\hat{e}) \in G.\omega(\tilde{e})$. As G acts freely, there is a unique $g_{\tilde{e}} \in G$ with $v(\tilde{e}) = g_{\tilde{e}}\,\omega(\tilde{e})$.

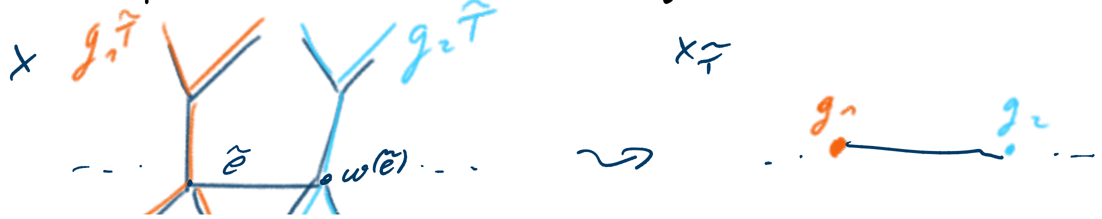


Claim: G is free with basis $\{g_{\tilde{e}} \mid \tilde{e} \in \tilde{E}\} =: S$.

Pf of claim:

The $g\tilde{T}$, $g \in G$ are p.w. disjoint subtrees and $X^0 = \bigcup_{g \in G} (g\tilde{T})^0$.

Define $X_{\tilde{T}}$ as the tree obtained by collapsing all $g\tilde{T}$.



The vertices of $X_{\tilde T}$ are identified with the elements of G.
There is an edge from g_1 to g_2 in $X_{\tilde T}$ iff
" " " " " $g_1\tilde T$ " $g_2\tilde T$ in X iff

$g_1^{-1} g_2 \in S$.
It follows that $X_{\tilde T}$ is isomorphic to the Cayley graph $\Gamma(G,S)$. The result now follows from Lemma 4.4. $\square$

Cor 4.7 (Nielsen-Schreier):
1) Every subgroup of a free group is free.
2) If G is free of finite rank and H is a subgroup of index m, then

$$\mathrm{rk}(H) = m\cdot(\mathrm{rk}(G)-1)+1.$$

Pf.:
1) Let $H \leq F(S)$. Then by Cor 4.5, $F(S)$ acts freely and without inversion on a tree X. But then H acts acts freely and without inversions on X. So it is free by Thm 4.6.
2) Let S be a basis G and $H\backslash G$ the set of right cosets of H in G. The group H acts $\Gamma(G,S)$ by

$$h.g := hg$$
$$h.(g,s) := (hg,s).$$

Hence, the quotient graph $Y := {}_H\backslash^{\Gamma(G,S)}$ has vertices $Y^0 = H\backslash G$ and pos. orient edges $Y^1_+ = (H\backslash G)\times S$, where

$Y^0 = H\backslash G$ and pos. orient edges $Y_+^1 = (H\backslash G) \times S$, where $\alpha(Hg, s) = Hg$, $\omega(Hg, s) = Hgs$. By Thm 4.6, we have

$$\begin{aligned} rk(H) &= |Y_+^1| - |Y^0| + 1 \\ &= |H\backslash G| \cdot |S| - |H\backslash G| + 1 \\ &= m \cdot rk(G) - m + 1 \end{aligned}$$

□

5. Free products and playing Ping-Pong

Def 5.1: Let A, B be groups. Without loss, assume that $A \cap B = \{1\}$.

- A normal form is an expression of the form $g_1 g_2 \cdots g_n$, where $n \geq 0$, $g_i \in (A \cup B) \setminus \{1\}$ and $g_i \in A$ iff $g_{i+1} \in B$. The number n is the length of the normal form and we identify the normal form of length 0 with 1.
- We define a multiplication on the set of normal forms as follows via induction over the length:
  - i) $1 \cdot x = x \cdot 1 = x$
  - ii) If $x = g_1 \cdots g_n$ and $y = h_1 \cdots h_m$ are normal forms with $n, m \geq 1$, then

$$x \cdot y := \begin{cases} g_1 \cdots g_n h_1 \cdots h_m & \text{if } g_n \in A, h_1 \in B \text{ or } g_n \in B, h_1 \in A \\ g_1 \cdots g_{n-1} z h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or } g_n, h_1 \in B \text{ and } g_n h_1 = z \neq 1 \\ g_1 \cdots g_{n-1} \cdot h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or } g_n, h_1 \in B \text{ and } g_n h_1 = 1 \end{cases}$$

induction →

- This multiplication makes the set of normal forms into a group, the free product $A * B$.

$(g_1 \cdots g_n)^{-1} = (g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1})$

Rem 5.2: A and B are naturally embedded in $A * B$.

Prop 5.3: If A, B are subgroups of G s.t. every $g \in G$ can be uniquely written as a product $g = g_1 \cdots g_n$ where $\forall i$, $g_i \in (A \cup B) \setminus \{1\}$ and $g_i \in A$ iff ${}_{i+1} \in B$, then $G \cong A * B$.