5. Free products and playing Ping-Pong

Def 5.1: Let A, B be groups. Without loss, assume that $A \cap B = \{1\}$.

- A <u>normal form</u> is an expression of the form $g_1 g_2 \cdots g_n$, where $n \geq 0$, $g_i \in (A \cup B) \setminus \{1\}$ and $g_i \in A$ iff $g_{i+1} \in B$. The number n is the <u>length</u> of the normal form and we identify the normal form of length 0 with 1.
- We define a multiplication on the set of normal forms as follows via induction over the length:
  i) $1 \cdot x = x \cdot 1 = x$
  ii) If $x = g_1 \cdots g_n$ and $y = h_1 \cdots h_m$ are normal forms with $n, m \geq 1$, then

$$x \cdot y := \begin{cases} g_1 \cdots g_n h_1 \cdots h_m, & \text{if } g_n \in A, h_1 \in B \text{ or } g_n \in B, h_1 \in A \\ g_1 \cdots g_{n-1} z h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or } g_n, h_1 \in B \text{ and } g_n h_1 = z \neq 1 \\ g_1 \cdots g_{n-1} \cdot h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or } g_n, h_1 \in B \text{ and } g_n h_1 = 1 \end{cases}$$

induction $\Rightarrow$

- This multiplication makes the set of normal forms into a group, the <u>free product</u> $A * B$.

Rem 5.2: A and B are naturally embedded in $A * B$.

Ex.: $F(\{a,b\}) = \langle a \rangle * \langle b \rangle \cong \mathbb{Z} * \mathbb{Z}$, more general

$F_n \cong F_{n-1} * \mathbb{Z}$.
$\leftarrow$ free, n generators

Prop 5.3: If A, B are subgroups of G s.t. every $g \in G$ can be uniquely written as a product $g = g_1 \cdots g_n$, where $\forall i$, $g_i \in (A \cup B) \setminus \{1\}$ and $g_i \in A$ iff $g_{i+1} \in B$, then $G \cong A * B$.

Cor 5.4: If $G = \langle A, B \rangle$ and for every non-triv. normal form $g = a_1 b_1 \cdots a_k (\cdot b_k)$, one has $g \neq 1$, then $G \cong A * B$.

Proof of Cor. 5.4: see padlet exercise 2 and explanations in the video recording.

Rem 5.5 (Torsion & centralisers in free products)

i) If $g = g_1 \cdots g_{2m}$, $m \geq 1$ is a normal form for $A * B$, then g has infinite order.

$\rightsquigarrow$ Every element of finite order in $A*B$ is conjugate to an elt. in A or B. In particular, a free product of torsion-free groups is torsion-free.

ii) If $G = A * B$ and $g = ab$, where $a \in A \setminus \{1\}$, $b \in B \setminus \{1\}$, then for the centraliser of g, we have

$$C_G(g) := \{ h \in G \mid hg = gh \} = \{ g^k \mid k \in \mathbb{Z} \} \cong \mathbb{Z}$$

$$\rightsquigarrow Z(G) := \{ h \in G \mid hg = gh \ \forall g \in G \} = \{1\}.$$

Rem 5.6:

i) If $A = \langle X \mid R \rangle$, $B = \langle Y \mid S \rangle$ with $X \cap Y = \emptyset$, then $A * B \cong \langle X \cup Y \mid R \cup S \rangle$.

ii) The free product is the coproduct in the category of groups. I.e., it can be defined via the following universal property:

Given homomorphisms $\varphi_A : A \to G$ and $\varphi_B : B \to G$, there is a unique hom. $\varphi : A * B \to G$ that makes the following diagram commute:

$$\begin{tikzcd}
A * B \arrow[dr, dashed, "\exists! \varphi"] & A \arrow[l, hook'] \arrow[d, "\varphi_A"] \\
B \arrow[u, hook] \arrow[r, "\varphi_B"'] & G
\end{tikzcd}$$

Thm 5.7: No group can be written both as a non-trivial free product and a non-trivial direct product.

Pf.: Let $G = A * B$, $A, B \neq \{1\}$ and take $a \in A \setminus \{1\}$ and $b \in B \setminus \{1\}$. Let $g := ab$. Then $C_G(g) \cong \mathbb{Z}$.

Now assume that $G = D \times E$, where $D, E \neq \{1\}$. Then g can be written as $g = d \cdot e$, $d \in D$, $e \in E$. We have

$$C_G(g) = C_D(d) \times C_E(e)$$

As $D, E \neq \{1\}$, we have $C_D(d) \neq \{1\} \neq C_E(e)$ so $C_G(g) \cong \mathbb{Z}$ ↯

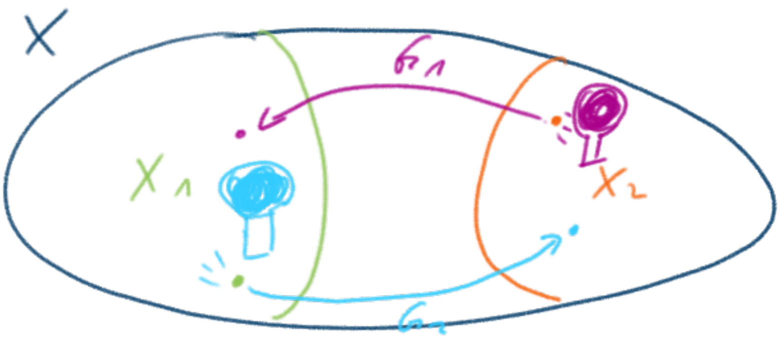
□

Lemma 5.8 (Ping-Pong): Let G be a group that acts on a set X and let $G_1, G_2 \leq G$ be subgroups with $|G_1| \geq 3$, $|G_2| \geq 2$.

If there are non-empty $X_1, X_2 \subseteq X$, with $X_2 \not\subseteq X_1$ such that

$$g(X_2) \subseteq X_1 \quad \forall g \in G_1 \setminus \{1\} \text{ and}$$
$$g(X_1) \subseteq X_2 \quad \forall g \in G_2 \setminus \{1\},$$

then $H := \langle G_1, G_2 \rangle$ is isomorphic to $G_1 * G_2$.



Pf.: Let w be a word in non-triv. normal form with resp. to G_1, G_2.

By Cor 5.4, need to show that $w \neq 1$.

Case 1: w starts and ends with a letter from G_1, i.e.

$$w = a_1 b_1 \cdots b_{k-1} a_k \quad \text{with } a_i \in G_1 \setminus \{1\},\ b_j \in G_2 \setminus \{1\}.$$

We have

$$w(X_2) = a_1 b_1 \cdots a_k (X_2) \subseteq a_1 b_1 \cdots b_{k-1}(X_1) \subseteq \dots$$
$$\dots \subseteq a_1 (X_2) \subseteq X_1,$$

so $w \neq 1$.

Case 2: $w = a_1 b_1 \dots a_k b_k$ is a word in normal form ending with $b_k \in G_2 \setminus \{1\}$. Then take $a \in G_1 \setminus \{1, a_1\}$ and con-

Case 2: $w = a_1 b_1 \dots a_k b_k$ [illegible]
with $b_k \in G_2 \setminus \{1\}$. Then take $a \in G_1 \setminus \{1, a_1\}$ and consider $a^{-1} w a = (a^{-1} a_1) b_1 \dots b_k \cdot a$

By case 1, $a^{-1} w a \neq 1$, hence so is w $\square$

Example: $SL_2(\mathbb{Z})$

Let $SL_2(\mathbb{Z})$ be the group of invertible 2×2-matrices over $\mathbb{Z}$ with det. 1. Its center is

$$Z(SL_2(\mathbb{Z})) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z}) / Z(SL_2(\mathbb{Z}))$$

Thm 5.9: $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

Pf.: Let $A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $B := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2 \mathbb{Z}$.

Then $\langle A, B \rangle = SL_2 \mathbb{Z}$.

Let $\varphi : SL_2 \mathbb{Z} \to PSL_2 \mathbb{Z}$ be the projection and $\alpha := \varphi(A)$, $\beta := \varphi(B)$. Clearly $\langle \alpha, \beta \rangle = PSL_2 \mathbb{Z}$.

Also, we have $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = B^3$, hence $\alpha^2 = 1_{PSL_2 \mathbb{Z}} = \beta^3$.

To apply the Ping-Pong Lemma, consider the action of $PSL_2 \mathbb{Z}$ on $\mathbb{R} \setminus \mathbb{Q}$.

$SL_2 \mathbb{Z} \curvearrowright \mathbb{R} \setminus \mathbb{Q}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . z := \frac{az+b}{cz+d}.$$

($cz + d \neq 0$ for $c, d \in \mathbb{Z}$, z irrational).

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ is in the kernel of this action,

$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right\}$ is in the kernel of this action, so we get an induced action $PSL_2\mathbb{Z} \curvearrowright \mathbb{R}\setminus\mathbb{Q}$.

We have

$$\alpha . z = \frac{1}{-z} = -\frac{1}{z}$$

$$\beta . z = \frac{z-1}{z} = 1 - \frac{1}{z} \quad , \beta^2 . z = \frac{1}{1-z}$$

Now let $X_1 := \mathbb{R}_{>0}\setminus\mathbb{Q}$, $X_2 := \mathbb{R}_{<0}\setminus\mathbb{Q}$.

Then $\langle\alpha\rangle . X_1 \subseteq X_2$ and $\langle\beta\rangle . X_2 \subseteq X_1$. Hence the claim follows from the P-P-Lemma.

$$\Big(PSL_2\mathbb{Z} \cong \underbrace{\langle\alpha\rangle}_{\mathbb{Z}/2\mathbb{Z}} * \underbrace{\langle\beta\rangle}_{\mathbb{Z}/3\mathbb{Z}} \Big)$$

$\square$

Aside

Some explanations on where the action in the previous proof comes from; this is an aside and the explanations were a bit hand-waving. If you are interested, feel encouraged to look up further details.

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} . z := \frac{az+b}{cz+d} \quad \text{extends to an action}$$

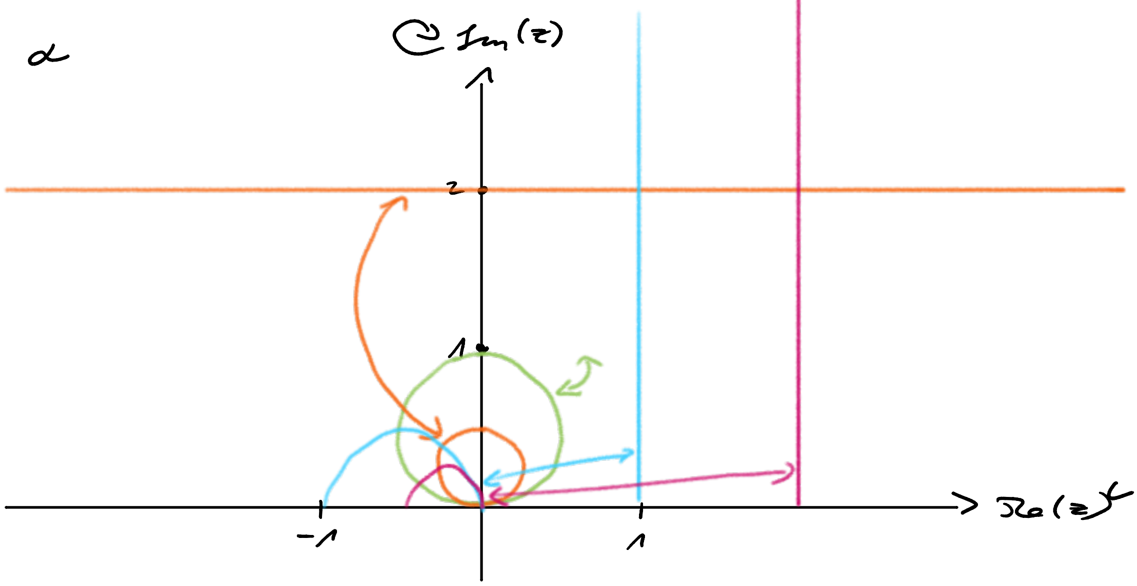
$$PSL_2\mathbb{Z} \curvearrowright \mathbb{C}\cup\{\infty\}$$

$\rightarrow$ Möbius transformation

Maps restrict to $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

$\alpha(z) = -\frac{1}{z}$

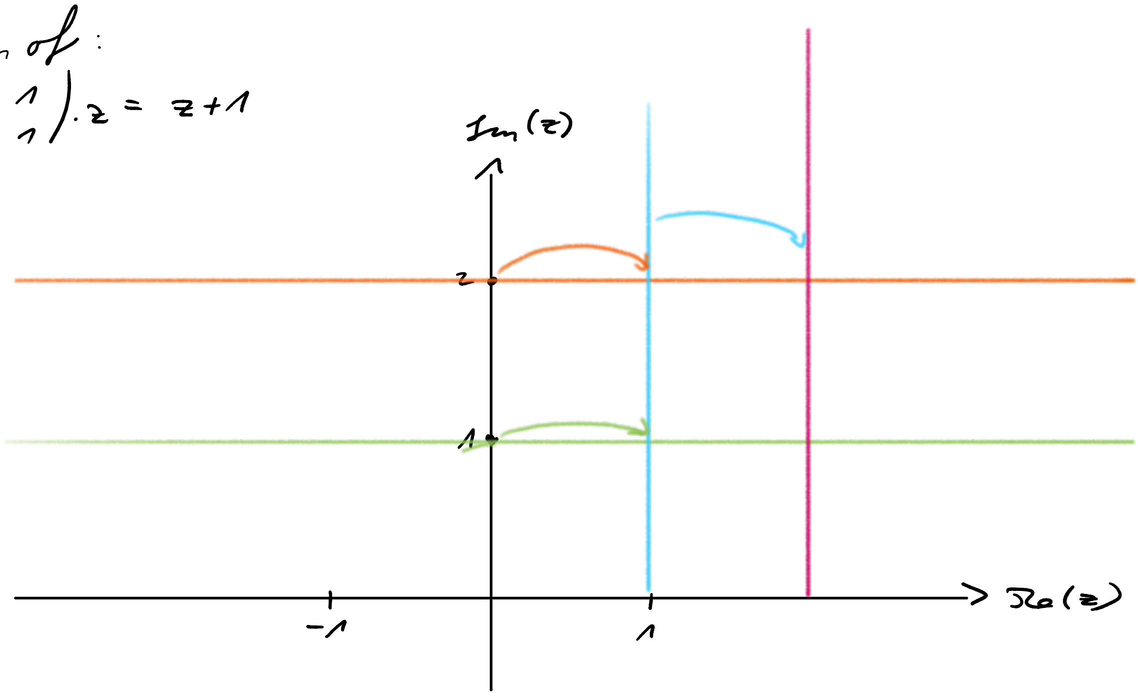
α



$\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \alpha$

Action of:

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . z = z+1$



$\rightsquigarrow$ isometries of hyperbolic plane $\mathbb{H}^2$

("upper halfplane model")