Lecture 6 - Script

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6. Amalgamated products

Def 6.1: Let G, H be groups, $A \leq G$, $B \leq H$ subgroups and $\varphi: A \to B$ an isomorphism. The group

$$G \underset{A=B}{*} H = G \underset{A}{*} H := \langle G * H \mid a = \varphi(a), a \in A \rangle$$

is called the <u>amalgamated product</u> of G and H over A.

$= G*H / \langle\langle \{\varphi(a)a^{-1} : a \in A\} \rangle\rangle^{G*H}$

Ex.: • $A = \{1\} \rightsquigarrow G \underset{A}{*} H = G * H$

• Seifert-van-Kampen Thm: X top. space, $U, V \subseteq X$ open, path-connected s.t.

• $X = U \cup V$

• $U \cap V \neq \emptyset$, path connected

Then $\pi_1(X) \cong \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V)$.

$\pi_1(U \cap V) \hookrightarrow \pi_1(U)$

$\hookrightarrow \pi_1(V)$

Def 6.2: Let J_A and J_B be systems of right coset representatives for A in G and B in H s.t. $1_G \in J_A$, $1_H \in J_B$ (<u>transversals</u>) (right coset: $A \cdot g$, $B \cdot g$)

An <u>A-normal form</u> in $G \underset{A}{*} H$ (wrt J_A, J_B) is a sequence $(x_0, \ldots, x_n)$, $n \geq 0$, where

i) $x_0 \in A$

ii) for $i > 0$, $x_i \in (J_A \setminus \{1_G\}) \cup (J_B \setminus \{1_H\})$ and $x_i \in J_A$ iff $x_{i+1} \in J_B$.

B-normal forms are defined analogously.

Ex.: $G = \langle a \mid a^{12} = 1 \rangle \cong \mathbb{Z}/12\mathbb{Z}$

$H = \langle b \mid b^{15} = 1 \rangle \cong \mathbb{Z}/15\mathbb{Z}$

$A = \langle a^4 \rangle \leq G \cong \mathbb{Z}/3\mathbb{Z}$, $B = \langle b^5 \rangle \leq H$ $\cong \mathbb{Z}/3\mathbb{Z}$

$f: A \to B \quad ; \quad f(a^4) = b^5$

Possible choices for J_A, J_B:

$$A\backslash G = \{A \cdot 1, A \cdot a, A \cdot a^2, A \cdot a^3\}$$

$$J_A = \{1_G, a, a^2, a^3\} \qquad \text{alternative would be: } J_A = \{1_G, a^5, a^2, a^{11}\}$$

$$J_B = \{1, b, b^2, b^3, b^4\}$$

$$G \ast_{A=B} H = \langle a, b \mid a^{12} = 1, b^{15} = 1, a^4 = b^5 \rangle$$

Write f in normal form:

$$f = a^3 b a^5 = a^3 b a^4 \cdot \underset{J_A}{a} = a^3 b b^5 \cdot a = a^3 b^6 \cdot a = \ldots$$

$$\ldots = a^3 \underset{\in B}{b^5} \cdot \underset{J_B}{b} \cdot \underset{J_A}{a} = a^3 a^4 \cdot b \cdot a = a^7 b \cdot a = \underset{A}{a^4} \cdot \underset{J_A}{a^3} \cdot \underset{\in J_B}{b} \cdot \underset{J_A}{a}$$

$\to (a^4, a^3, b, a)$ is an A-normal form "for f".

Rem 6.3: There is a bijection from the set of A-normal forms to the set of B-normal forms given by

$$(x_0, x_1, \ldots, x_n) \mapsto (f(x_0), x_1, \ldots, x_n).$$

Ex $(a^4, a^3, b, a) \mapsto (b^5, a^3, b, a)$

Thm 6.4: Any element $f \in F := G \ast_A H$ can uniquely be written in normal form. I.e. for transversal J_A, J_B, there is a unique A-normal form $(x_0, \ldots, x_n)$ s.t. $f = x_0 \cdot \ldots \cdot x_n$.

Pf.: Existence as in example.

Uniqueness: Let W_A, W_B be the set of A- and B-normal forms and let $f_\ast : W_A \to W_B$ be the bij. from Rem 6.3.

Idea: Define an action $F \curvearrowright W_A$.

First define an action of $G \curvearrowright W_A$ as follows:

First define an action of $G \curvearrowright W_A$ as follows:

For $g \in G$, write $g = \tilde{g}\,\bar{g}$ for the unique decomp. with $\tilde{g} \in A$, $\bar{g} \in J_A$.

Ex.. above: $g = a^3 b a^5 = \underbrace{a^3 b a^4}_{\tilde{g}} \cdot \underbrace{a}_{\bar{g}}$

Now for $\tau = (x_0, \dots, x_n) \in W_A$, define:

if $n = 0$,

$$g \cdot \tau = g \cdot (x_0) = \begin{cases} (g x_0) & \text{, if } g \in A \\ (\widetilde{g x_0}, \overline{g x_0}) & \text{, if } g \notin A \end{cases}$$

if $n \geq 1$,

$$g \cdot \tau = \begin{cases} (g x_0, x_1, \dots, x_n) & \text{, if } g \in A \\ (\widetilde{g x_0}, \overline{g x_0}, x_1, \dots, x_n) & \text{, if } g \notin A, x_1 \in H \\ (g x_0 x_1, x_2, \dots, x_n) & \text{, if } g \notin A, x_1 \in G, g x_0 x_1 \in A \\ (\widetilde{g x_0 x_1}, \overline{g x_0 x_1}, x_2, \dots, x_n) & \text{, if } g \notin A, x_1 \in G, g x_0 x_1 \notin A \end{cases}$$

Exerc.: Check that the def. defines an action on W_A.

Analogously, define $H \curvearrowright W_B$. Use this to get an action $H \curvearrowright W_A$, $h \cdot \tau := f_*^{-1}(h \cdot f_*(\tau))$, for $h \in H$, $\tau \in W_A$.

For all $a \in A$, $\tau \in W_A$, we have $a \cdot \tau = f(a) \cdot \tau$, so we get an induced action $F = G *_A H \curvearrowright W_A$. See Padlet.

To finish, let $f \in F$ and write it in A-norm. $f = x_0 x_1 \dots x_n$.

Then the the image of $(1) \in W_A$ under the action of f is:

$$f \cdot (1) = x_0 \dots x_n (1) = x_0 \dots x_{n-1} (1, x_n) = x_0 \dots x_{n-2} (1, x_{n-1}, x_n) = \dots$$

$$\dots = x_0 (1, x_1, \dots, x_n) = (x_0, x_1, \dots, x_n).$$

Hence, the image of (1) uniquely determines the normal form decomp. of f. $\square$

Cor 6.5: Let $F = G *_A H$. The canonical projection $G * H \overset{i}{\twoheadrightarrow} F$

Cor 6.5: Let $F = G *_A H$. The canonical projection $G * H \overset{\iota}{\twoheadrightarrow} F$ induces embeddings $G, H \to F$. The images $\iota(G), \iota(H)$ of these embeddings generate F and $\iota(G) \cap \iota(H) = \iota(A) = \iota(B)$.

Cor. 6.6: If F is a group and $G, H \leq F$, $A \leq G \cap H$ are subgroups such that every $f \in F$ has a unique A-normal form, then $F \cong G *_A H$.

Rem 6.7: The amalgamated product is the cofibre product in the category of groups. I.e., it can be defined via the following universal property:
Let $A \overset{\iota_G}{\hookrightarrow} G$, $A \overset{\iota_H}{\hookrightarrow} H$ be inj. homomorphisms. Then for every pair of homomorphisms $\varphi_G : G \to K$, $\varphi_H : H \to K$ s.t. $\varphi_G \circ \iota_G = \varphi_H \circ \iota_H$, there is a unique homomorphism $\varphi : G \underset{\iota_G(A) = \iota_H(A)}{*} H \to K$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, hook, bend left, "\iota_G"] \arrow[ddr, hook, bend right, "\iota_H"'] & & \\
& G \underset{\iota_G(A)=\iota_H(A)}{*} H \arrow[dr, dashed, "\exists!\varphi"] & G \arrow[l] \arrow[d, "\varphi_G"] \\
& H \arrow[u, hook] \arrow[r, "\varphi_H"] & K
\end{tikzcd}$$

Def 6.8: A <u>segment</u> is a graph consisting of two vertices and two mutually inverse edges between them.

$\bullet \xrightarrow{e} \bullet$

Rem 6.9: $G \curvearrowright X$ graph without inversions s.t. $G \backslash X$ is a segment:

$\rightsquigarrow$ $G \curvearrowright X^0$ has two orbits $X^0 = X^0_1 \,\dot\cup\, X^0_2$, $G \curvearrowright X^0_i$ trans.

$G \curvearrowright X^1_+$ transitive

$\rightarrow$ get an induced colouring of vertices and X is bipartite, i.e. there are no edges between vertices

bipartite, i.e. there are no edges between vertices of the same colour
(i.e. if $v, v' \in X_1^0$, then there is no edge between v and v')
(Converse true as well.)

Def 6.10 (ad-hoc): Let G be a group, G_1, G_2 subgroups and $A \leq G_1 \cap G_2$. The associated <u>coset graph</u> is the graph defined by

$$X^0 := G/G_1 \cup G/G_2 \quad , \quad X_+^1 := G/A \quad \text{with}$$

$$\alpha(gA) = gG_1 \quad , \quad \omega(gA) = gG_2 .$$

Observation 6.11.

1) The edges starting at $g \cdot G_1$ are all of the form $g \cdot g_1 A$, where $g_1 \in G_1$. Analogously for $g \cdot G_2$.
This implies: Let $\overline{J}_A$ and $\overline{J}_B$ be transversal for the sets of <u>left</u> cosets G_1/A and G_2/A. Let $g \in G$. Then...
... for every $g_1 \in \overline{J}_A$, there is exactly one positively oriented edge that starts at $g \cdot G_1$. It is of the form $g \cdot g_1 A$
... for every $g_2 \in \overline{J}_B$, there is exactly one positively oriented edge that ends at $g \cdot G_2$. It is of the form $g \cdot g_2 A$.

2) Let J_A and J_B be transversals for the sets of <u>right</u> cosets $A \backslash G_1$ and $A \backslash G_2$.
There is a 1-to-1 correspondence

$$\{A\text{-normal forms of } g \text{ with resp. to } J_A, J_B\}$$
$$\cong$$
$$\{\text{reduced paths in } X \text{ from } 1 \cdot G_1 \text{ to } g^{-1} \cdot G_1\}.$$

Pf.: 1) exercise
2) Using the bijection between right and left cosets given by

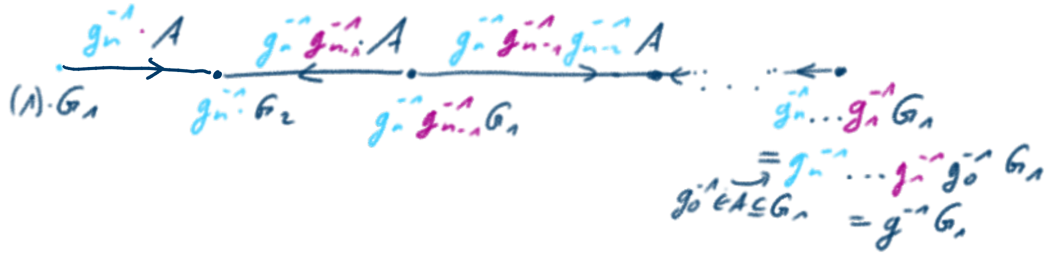
$$A \backslash G_i \longrightarrow G_i / A$$
$$A g_i \longmapsto g_i^{-1} A ,$$

$$Ag_i \longmapsto g_i^{-1}A,$$

the transversals J_A, J_B give us transversals $\overline{J_A}, \overline{J_B}$ for the left cosets of A.

Let $g = g_0 \cdots g_n$ be a normal form for g, i.e. $g_0 \in A$ and the g_i are alternating between J_A and J_B. Assume that $g_1 \in J_B$, $g_n \in J_A$.

Then $g^{-1} = g_n^{-1} \cdots g_1^{-1} g_0^{-1}$, where $g_0^{-1} \in A$ and the g_i^{-1} are alternating between $\overline{J_A}$ and $\overline{J_B}$. By 1), there is a unique reduced path of the form



If $g_n \in J_B$ or $g_1 \in J_A$, concatenate the path with the (unique) edges $1\cdot G_1 \longrightarrow 1\cdot G_2$ or $g_n^{-1} \cdots g_1^{-1} G_2 \longleftarrow g_n^{-1} \cdots g_1^{-1} G_1$.

On the other hand, 1) implies that every path from $1 \cdot G_1$ to $g^{-1} G_1$ can be written in the form above. Hence, it defines a unique A-normal form for g. $\square$