6. Amalgamated products (cont.)

Thm 6.12: Let $G = G_1 *_A G_2$. Then there is a tree X s.t. G acts on X without inversions and such that the quotient graph $G\backslash X$ is a segment. Moreover, this segment has a lift to a segment $\overset{\alpha(e)}{\bullet}\xrightarrow{e}\overset{\omega(e)}{\bullet}$ in X s.t. $G_e = A$, $G_{\alpha(e)} = G_1$ and $G_{\omega(e)} = G_2$.

Pf.: Let X be as in Def 6.10

G acts on X by left mult. Cond of Lem 6.9 are satisfied

$\rightsquigarrow$ quotient $G\backslash X$ is a segment.

Let $\tilde{T}$ be the segment $\underset{1\cdot G_1}{\bullet}\xrightarrow{1\cdot A}\underset{1\cdot G_2}{\bullet}$. Then $\tilde{T}$ is a lift of $G\backslash X$ with the desired stabilisers.

Need to show: X is a tree

Step 1: X is connected (existence of normal forms)

It is sufficient to show that for $g \in G$, there is a path from $1\cdot G_1$ to $g\cdot G_1$

This follows from the existence of normal forms and Obs. 6.11.2.

Step 2: X has no circuit (uniqueness of normal forms)

Assume there is a closed reduced path in X. By Lem 6.9, X is bipartite so its length n must be even. As G has only two orbits of vertices, there must be a closed reduced path $e_1, \dots, e_n$ with $\alpha(e_1) = 1\cdot G_1$.

It follows that there are two distinct red. paths from $1\cdot G_1$ to $x := \omega(e_{\frac{n}{2}})$.

We have $x = g^{-1} G_i$ for some $g \in G$, $i \in \{1,2\}$. By Obs. 6.11.2, this means that there are two dist. normal forms for g. But normal forms in amalg. are unique, so we get a contrad. $\square$

Thm 6.13: Let G be a group that acts on a tree X s. t. the quotient $G \backslash X$ is a segment and let $\widetilde{T} = \overset{\alpha(e)}{\bullet} \xrightarrow{e} \overset{\omega(e)}{\bullet} \subseteq X$ be a lift of this segment. Then $G \cong G_{\alpha(e)} \underset{G_e}{*} G_{\omega(e)}$.

Pf.: Let $G_1 := G_{\alpha(e)}$, $G_2 := G_{\omega(e)}$, $A := G_e$.

We can identify the vertex and edge sets of X with

$$X^0 \hat{=} G/G_1 \cup G/G_2, \quad X^1_+ = G/A.$$

Let Y be the coset graph def. in Def 6.10

Claim: The map

$$X \xrightarrow{\psi} Y$$
$$g.\alpha(e) \mapsto gG_1$$
$$g.\omega(e) \mapsto gG_2$$
$$g.e \mapsto g.A$$

defines an isomorphism of graphs.

Pf.: Bij. on vertex and edge sets is clear.

We only need to check that adj. is preserved But this is obvious:

$$\psi(\alpha(g.e)) = \psi(g.\alpha(e)) = gG_1 = \alpha(gA) = \alpha(\psi(g.e)).$$

same for ω.

This implies that the coset graph is a tree, i.e. there is a [illegible] ... By Obs 6.11.2, This implies that the coset graph is a tree, i.e. there is a unique red. path between any two vertices. By Ob. 6.11.2, this implies that every elt. of G has a unique A-normal form. Now apply Cor 6.6. $\square$

7. HNN extensions

Def 7.1: Let G be a group, $A, B \leq G$, $\varphi: A \to B$ an isomorphism and $\langle t \rangle = \mathbb{Z}$ the infinite cyclic group generated by t.

- The HNN extension of G with respect to A, B and φ is the group

$$G^* = G*_\varphi = \langle G * \langle t \rangle \mid t^{-1} a t = \varphi(a),\ a \in A \rangle$$

$$= G * \langle t \rangle / \langle\langle t^{-1} a t\, \varphi(a)^{-1} \mid a \in A \rangle\rangle^{G * \langle t \rangle}$$

- G is called the base, t the stable letter and A and B are the associated subgroups.

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Ex.: Semidir. prod. $G \rtimes_\varphi \mathbb{Z} = \langle t \rangle$

$A = G = B \quad \varphi: A \to B \in \operatorname{Aut}(G)$

Elt. in $G \rtimes_\varphi \mathbb{Z}$: (g, t^k)

$$t^{-2} g t^3 = t^{-1}\, t^{-1} g t\, t^3 = t^{-1} \varphi(g) t^4$$
$$= \varphi^2(g) t^5$$

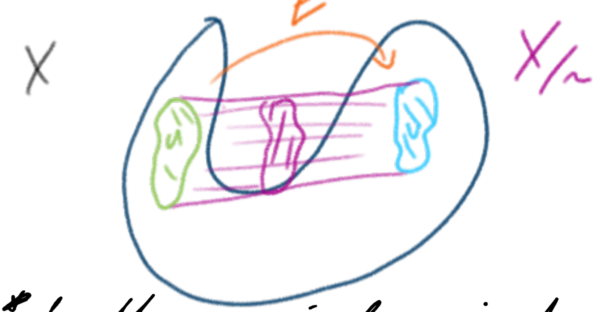
in HNN: $t^{-1} g t = \varphi(g)$

$t^{-1} \varphi(g) t = \varphi(\varphi(g))$

etc.

- Thm: X top., U, V connected spaces, $f: U \to V$ homeo.
  Then $\pi_1(X / \{u \sim f(u)\})$ is an HNN extension of $\pi_1(X)$

Then $\pi_1(X/\{u\sim f(u)\})$ is an HNN extension of $\pi_1(X)$ with assoc. subgr. the images of $\pi_1(U)$ and $\pi_1(V)$ in $\pi_1(X)$.



Let $\iota : G * \langle t \rangle \to G^*$ be the canonical proj. Any $x \in G^*$ can be written as

$$x = \iota(g_0)\,\iota(t)^{\varepsilon_1}\,\iota(g_1) \dots \iota(t)^{\varepsilon_n}\,\iota(g_n),$$

where $g_i \in G$, $\varepsilon_j \in \{\pm 1\}$. To shorten notation, we write

$$x = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n.$$

Def 7.2: Let J_A, J_B be transversals of $A\backslash G$, $B\backslash G$ s.t. $1 \in J_A, J_B$. For $g \in G$, let $\bar{g} \in J_A$, $\hat{g} \in J_B$ be the unique elements with $Ag = A\bar{g}$, $Bg = B\hat{g}$.

- A <u>normal form</u> for G^* (wrt J_A and J_B) is a sequence $(g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n)$ such that
  - i) $g_0 \in G$;
  - ii) $\varepsilon_i = -1$ iff $g_i \in J_A$ and $\varepsilon_i = +1$ iff $g_i \in J_B$;
  - iii) there is no consecutive subsequence of the form $t^{\varepsilon}, 1, t^{-\varepsilon}$.

Ex.: $G^* = \langle a, b, t \mid t^{-1} a^2 t = b^3 \rangle$ be the HNN ext of $G = F(a,b)$ with assoc. subgr. $A = \langle a^2 \rangle$ and $B = \langle b^3 \rangle$.

$J_A :=$ set of all red. words in $F(a,b)$ that don't begin with a power of a other a^1

$J_B :=$ " . . . — "

with a power of b other b^1 or b^2.

Let $x = b^2 t^{-1} a^{-4} t b^5 a b \underbrace{t^{-1} a^4 b^3 a}_{= \underbrace{t^{-1} a^4}\, \cdot\, b^3 a}$. Write in normal form:

$x = b^2 t^{-1} a^{-4} t b^5 a b^7 t^{-1} b^3$ | $t^{-1} a^4 t = b^6$

form:

$$= \underbrace{t^{-1}a^4}\cdot b^3 a$$

$$x = b^2 t^{-1} a^{-4} t\, \underbrace{b^5 a b^7}\, t^{-1} \underbrace{b^3}_{J_A} a$$

$t^{-1}a^4 t = b^6$
$\Leftrightarrow t^{-1}a^4 = b^6 t^{-1}$

$$= b^2\, \underbrace{t^{-1} a^{-2}\, t}\cdot \underbrace{b^2 a b^7}_{J_B}\, t^{-1} b^3 a$$

$t^{-1}a^2 t = b^3 \Leftrightarrow t b^3 = a^2 t$

$$= b^2 b^{-3} b^2 a b^7 t^{-1} b^3 a$$

$$= \underbrace{bab^7}_{\in G}\, t^{-1} \underbrace{b^3 a}_{\in J_A}.$$

Thm 7.3: Let $G^* = \langle G, t \mid t^{-1} a t = \varphi(a), a \in A\rangle$ be an HNN-extension with associated subgroups $A \xrightarrow{\varphi} B$. Then

1) Every $x \in G^*$ can be uniquely written in normal form, i.e. for transversal J_A, J_B, there is a unique normal form $(g_0, t^{\varepsilon_1}, \dots, t^{\varepsilon_n}, g_n)$ such that $x = g_0 t^{\varepsilon_1} \dots t^{\varepsilon_n} g_n$.

2) The map $G \to G^*$, $g \mapsto g$ is a injective (hence $G \hookrightarrow G^*$ canonically).

If $w = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$, where $n \geq 1$ and there are no subwords $t^{-1} g_i t$, $g_i \in A \cup B$, then $w \neq 1$.

Pf.:

1) existence as in example

Uniqueness: Similar to amalg. prod., define action $G^* \curvearrowright W$ set of normal forms. $g.(1)$ will be normal form of g.

Let $\tau = (g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n) \in W$. The actions of $g \in G$, t and t^{-1} on τ are defined as follows:

$$g \cdot \tau = (gg_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n); \quad (\mathrm{I})$$

$$t \cdot \tau = \begin{cases} (\varphi^{-1}(g_0)g_1, t^{\varepsilon_2}, g_2, \dots, t^{\varepsilon_n}, g_n) & \text{if } \varepsilon_1 = -1, g_0 \in B, \\ (\varphi^{-1}(b), t, \hat{g}_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n) & \text{otherwise,} \end{cases}$$

where b is the element of B such that $g_0 = b\hat{g}_0$;

$$t^{-1} \cdot \tau = \begin{cases} (\varphi(g_0)g_1, t^{\varepsilon_2}, g_2, \dots, t^{\varepsilon_n}, g_n) & \text{if } \varepsilon_1 = 1, g_0 \in A, \\ (\varphi(a), t^{-1}, \bar{g}_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n) & \text{otherwise,} \end{cases} \quad (\mathrm{II})$$

where a is the element of A such that $g_0 = a\bar{g}_0$.

(I) defines an action $G \curvearrowright W$

(II) " " " $\langle t \rangle \curvearrowright W$

To see that we get an induced $G^* = G * \langle t \rangle / N \curvearrowright W$ by checking that every elt of

$$N = \{ t^{-1} a t f(a)^{-1} \mid a \in A \}^{G * \langle t \rangle}$$

acts trivially. Now as in the case of amalg., verify that if g has normal form τ, then $g \cdot (1) = \tau$. 1) follows.

For 2) note every elt $g \in G$ acts non-triv. on W. Hence $G \cap N = \{1\}$, so G embeds in $G^* = G * \langle t \rangle / N$.

If w is as in the claim, we can rewrite it in normal form (as in example) and see that the normal form has length $2n+1 > 1$ and 1) implies that $w \neq 1$. $\square$