Lecture 8 - Script

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$= G*\langle t\rangle / \{t^{-1}at = f(a), a\in A\}$

Cor. 7.4: Let $G^* = \langle G, t \mid t^{-1}at = f(a), a \in A\rangle$ be an HNN-extension with associated subgroups $A \xrightarrow{f} B$. Then the canonical projection $c: G * \langle t\rangle \to G^*$ restricts to isomorphisms on G and $\langle t\rangle$. After identifying these groups with their images, A and B are conjugate in G^* under t.

Cor 7.5: Let H be a group with subgroups G and $\langle t\rangle \cong \mathbb{Z}$, let $A, B \le H$ such that $t^{-1}At = B$. If every element of H has a unique normal form, then $H \cong G^*$.

Prop 7.6: Let $H, K \le G$ and $f: H \to K$ an isomorphism. Then the HNN-extension $G^* = \langle G, t \mid t^{-1}ht = f(h)\rangle$ can be seen as a subgroup of an amalgamated product.

Pf.: Let $\langle u\rangle \cong \langle v\rangle \cong \mathbb{Z}$ and define

$X := G * \langle u\rangle$, $Y := G * \langle v\rangle$

$L := \langle G, u^{-1}Hu\rangle \le X$, $M := \langle G, v^{-1}Kv\rangle \le Y$

We have $L \cong G * u^{-1}Hu$, $M \cong G * v^{-1}Kv$. (We have

$g_1 \underbrace{u^{-1}h_1u} g_2 \underbrace{u^{-1}h_2u} g_3 \cdots u^{-1}h_nu \neq 1$.)

Define an isomorphism

$\tilde{f}: L \to M$

$g \mapsto g$ for $g \in G$

$u^{-1}hu \mapsto v^{-1}f(h)v$ for $h \in H$.

Let $\tilde{G} := X \underset{L=M}{*} Y$. Then $G \le \tilde{G}$ and for $h \in H$, we have

$u^{-1}hu = \tilde{f}(u^{-1}hu) = v^{-1}f(h)v$, i.e.

$u^{-1} h u = \overset{L=M}{\varphi}(u^{-1} h u) = v^{-1} \varphi(h) v$, i.e.

$\varphi(h) = \underbrace{(uv^{-1})^{-1}}_{= v u^{-1}} h \, (uv^{-1})$.

Setting $t := uv^{-1}$, we get

$$G^* = \langle G, t \mid t^{-1} h t = \varphi(h),\ h \in H \rangle \leq \hat{G}. \qquad \square$$

Thm 7.7 (HNN '49): Every countable group C can be embedded in a group G that is generated by two elements of infinite order and such that:

i) G has n-torsion iff C has n-torsion

ii) If C is finitely presented, so is G.

Pf.: Let $C = \langle c_1, c_2, \dots \mid r_1, r_2, \dots \rangle$ be a presentation of C with countable generating set.

Let $F := C * \underbrace{\langle a, b \rangle}_{= F(a,b)}$. The sets

$A_0 := \{a, b^{-1}ab, b^{-2}ab^2, \dots\}$

$B_0 := \{b, c_1 b^{-1}ab, c_2 b^{-2}ab^2, \dots\}$

both generate free subgroups (of rank $\aleph_0$) in F. (Non-tr. words over A_0 or B_0 represent non-tr. elts. of F.)

In particular, they are canonically isomorphic and we can define an HNN-extension of F by

$$G := \langle F, t \mid t^{-1} a t = b,\ t^{-1} b^{-i} a b^{i} t = c_i b^{-i} a b^{i} \rangle.$$

Claim: G is as we wanted

2-generators: a, t finite presentation clear; torsion follows from Lyndon-Schupp - Combinatorial Group Theory , Theorem 2.4 at p. 185 $\square$

Cor. 7.8 (B. Neumann): There are $2^{\aleph_0}$ many non-

isomorphic 2-generated groups

Cor. 7.8 (B. Neumann): There are $2^{\aleph_0}$ many non-isomorphic 2-generator groups.

Pf.: Let $\mathbb{P}$ be the set of all primes. For $S \subseteq \mathbb{P}$, define $T_S := \bigoplus_{p \in S} \mathbb{Z}/p\mathbb{Z}$. Then T_S embeds in some 2-generator group G_S such that G_S has p-torsion iff $p \in S$. Hence if $S \neq S' \subseteq \mathbb{P}$, then $G_S \ncong G_{S'}$. $\square$

Thm 7.9 (HNN '49): Every countable group can be embedded into a group where all elements of the same order are conjugate.

Pf.: We first prove: If C is a countable group, it embeds into a group C^* where all elts <u>from C</u> that have the same order are conjugate.

To see this, let

$$\{(a_i, b_i) \mid i \in I, \ a_i, b_i \in C \text{ with } \langle a_i \rangle \cong \langle b_i \rangle\}$$

and define

$$C^* := \langle C, \{t_i \mid i \in I\} \mid t_i^{-1} a_i t_i = b_i \rangle.$$

For the proof of the theorem, set $G_0 := C$ and for $i > 0$, let $G_{i+1} := G_i^*$ as above. Then $C_\infty = \bigcup_{i=0}^{\infty} G_i$ satisfies the claim. $\square$

Thm 7.10: Every countable group C can be embedded into a countable, simple, divisible group.

simple: no non-tr. normal subgroups

divisible: $\forall c \in C \ \forall n \in \mathbb{N} \ \exists d \in C : d^n = c$

Pf.: Let $K = C \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \oplus \underset{\overset{\shortparallel}{\mathbb{Z}}}{\mathbb{Z}}$. By Thm 7.7, $K * \langle x \rangle$

can be embedded into a 2-generator group $U = \langle a,b \rangle$, where a,b have inf. order.

By Thm 7.9, U can be embedded into a countable group G where all elements of the same order are conjugate.

Claim: G is simple.

Pf.: Let $\{1\} \neq N \trianglelefteq G$ and $z' \in N \setminus \{1\}$. There is $z \in K$ with $\langle z \rangle \cong \langle z' \rangle$. As z and z' are conjugate, we have $z \in K \cap N$.

N normal: $\forall g \in G$: $g^{-1} N g = N$

Hence, $n := [x,z] = x z x^{-1} z^{-1} \in N$ and this elt has infinite order. It follows that n is conj. to a and b, so as N is normal, $\langle a,b \rangle = U \subseteq N$. This implies $N = G$.

Point here: K<U<N and K contains an element of any order; it follows that every element of G is conjugate to sth. in N. $\square$

Cor 7.11: There are $2^{\aleph_0}$ countable simple groups.

Pf.: A countable group contains only countably many 2-generator subgroups. But there are $2^{\aleph_0}$-many 2-generator groups (Cor. 7.8) and each can be embedded in some countable simple group (Thm 7.10).

(Use: If κ, λ are inf. cardinals, then $\kappa \cdot \lambda = \max\{\kappa, \lambda\}$.) $\square$