Def 7.12: A loop is a graph consisting of one vertex and two mutually inverse edges:

Rem 7.13: $G \curvearrowright X$ without inversions. Then $G\backslash X$ is a loop iff G acts trans. on X^0 and X^1_+.

Thm 7.14: Let $G = H*_A = \langle H, t \mid t^{-1} a t = \varphi(a) \ \forall a \in A \rangle$ be an HNN-extension. Then there is a tree X s.t. G acts on X without inversions and such that the quotient graph $G\backslash X$ is a loop. Moreover, there is a segment

$$\tilde{Y} = \underset{\alpha(e)}{\bullet} \xrightarrow{\ e\ } \underset{\omega(e)}{\bullet}$$

in X s.t. $G_e = A$, $G_{\alpha(e)} = H$ and $G_{\omega(e)} = tHt^{-1}$.

Pf.: Similar to proof of Thm 6.12.

Define a graph X via

$$X^0 := G/H \quad , \quad X^1_+ := G/A$$

$$\alpha(gA) = gH \quad , \quad \omega(gA) = gtH.$$

G acts on X by left multiplication, the action is trans. on X^0 and X^1_+. Furthermore, the segment

$$\tilde{Y} = \underset{1\cdot H}{\bullet} \xrightarrow{\ 1\cdot A\ } \underset{t\cdot H}{\bullet}$$

satisfies the conditions of the claim.

$$\left(\begin{array}{l} g\cdot A = A \Leftrightarrow g \in A \\ g\cdot tH = tH \Leftrightarrow t^{-1} g tH = H \\ \qquad \Leftrightarrow t^{-1} g t \in H \end{array} \right)$$

Left to show: X is a tree, i.e. it is
i) connected and has ii) no circuits. The proof is similar to the one of Thm 6.12 i) existence of normal forms
ii) uniqueness of " " . $\square$

Thm 7.15: Let G be a group that acts without inversions on a tree X s.t. the quotient $G\backslash X$ is a loop and let $Y = \bullet \xrightarrow{\ e\ }$ be a lift of $G\backslash X$

sions on a tree X s.t. the quotient $G\backslash X$ is a loop and let $Y = \bullet \xrightarrow{e} \bullet$ be a lift of $G\backslash X$.
Let $g \in G$ s.t. $g(\alpha(e)) = \omega(e)$. Let $\varphi : G_e \to g^{-1}G_e g$ be the isomorphism given by conjugation with g. Then $g^{-1}G_e g \leq G_{\alpha(e)}$ and the homomorphism
$$\langle G_{\alpha(e)}, t \mid t^{-1} a t = \varphi(a) \ \forall a \in G_e \rangle \to G$$
given by sending t to g is an isomorphism. I.e. G can be written as an HNN-extension $G \cong G_{\alpha(e)}^{*}$.

Pf.: Sim. to the proof of Thm 6.13.
As $G\backslash X$ is a loop, G acts trans. on X^0 and X^1_+ (Rem 7.13).
So we can identify X^0 with $G/G_{\alpha(e)}$ and X^1_+ with G/G_e. Use this to show that X is isomorphic to the graph defined in the proof of Thm 7.14. $\square$

8. Graphs of groups and general Bass-Serre theory

Def 8.1: A <u>graph of groups</u> (G, Y) consists of a connected graph Y, a <u>vertex group</u> G_v for each vertex $v \in Y^0$, an <u>edge group</u> G_e for each edge $e \in Y^1$, where $G_{\bar{e}} = G_e$, and monomorphisms $\{\alpha_e : G_e \to G_{\alpha(e)} \mid e \in Y^1\}$.
We also write $\omega_e : G_e \to G_{\omega(e)}$ for the monomorphism $\alpha_{\bar{e}} : G_{\bar{e}} = G_e \to G_{\alpha(\bar{e})} = G_{\omega(e)}$.

Def. 8.2: Let (G, Y) be a graph of groups
- We write $F(G, Y)$ for the group defined by
$$\left\langle G_v, t_e : v \in Y^0, e \in Y^1 \;\middle|\; \begin{array}{l} t_e t_{\bar{e}} = 1,\ t_e^{-1} \alpha_e(g) t_e = \alpha_{\bar{e}}(g); \\ e \in Y^1,\ g \in G_e \end{array} \right\rangle$$
- Let $P \in Y^0$. The <u>fundamental group</u> $\pi_1(G, Y, P)$ <u>of (G, Y) with respect to P</u> is the subgroup of $F(G, Y)$ consisting of all elements of the form
$g_0 t_{e_1} g_1 t_{e_2} \cdots t_{e_n} g_n$,

...ng of all elements of the form

$g_0 t_{e_1} g_1 t_{e_2} \cdots t_{e_n} g_n$,

where $e_1 e_2 \dots e_n$ is a closed path in Y starting at P, $g_0 \in G_P$ and $g_i \in G_{w(e_i)}$.

• Let $T \subseteq Y$ be a spanning tree. The <u>fundamental group $\pi_1(\mathbb{G}, Y, T)$ of $(\mathbb{G}, Y)$</u> with respect to T is the quotient of $F(\mathbb{G}, Y)$ given by factoring out the normal closure of $\{t_e \mid e \in T^1\}$, i.e.

$$\pi_1(\mathbb{G}, Y, T) := \langle F(\mathbb{G}, Y) \mid t_e = 1, e \in T^1 \rangle .$$

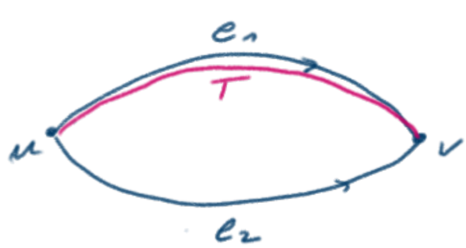
Ex.: $G_v = \{1\}\ \forall v \in Y^0 \rightsquigarrow \pi_1(\mathbb{G}, Y, P) \cong \pi_1(Y, P)$

Prop 8.3: See exercises on the padlet https://padlet.com/benjaminbrueckmaths/lecture9

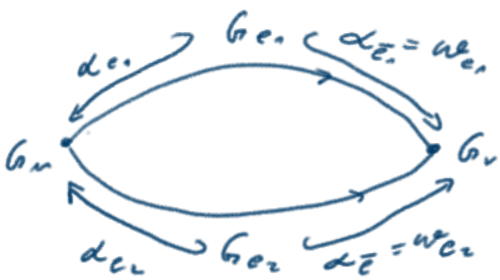
i) If $Y = \overset{P}{\bullet}\!\overset{e}{\text{———}}\!\overset{Q}{\bullet}$ is a segment, then the group $\pi_1(\mathbb{G}, Y, Y)$ is isomorphic to the free product of the groups G_P and G_Q amalgamated over the subgroups $\alpha_e(G_e)$ and $\alpha_{\bar{e}}(G_e)$.

ii) If $Y = P \bullet\!\bigcirc\ e$ is a loop, then the group $\pi_1(\mathbb{G}, Y, P)$ is isomorphic to the HNN extension with the base G_P and the associated subgroups $\alpha_e(G_e)$, and $\alpha_{\bar{e}}(G_e)$.

Sketch of solution to padlet exercise 3:



$T^1 = \{e_1, \bar{e}_1\}$



$$\pi_1(\mathbb{G}, Y, T) = \left\langle \boxed{G_u, G_v}, \cancel{t_{e_1}, t_{\bar{e}_1}}, t_{e_2}, t_{\bar{e}_2} \;\middle|\; \begin{array}{ll} \cancel{t_{e_1} t_{\bar{e}_1} = 1} = t_{e_2} t_{\bar{e}_2}, & \\ \cancel{t_{e_1}^{-1}} \boxed{\alpha_{e_1}(g) \cancel{t_{e_1}} = \omega_{e_1}(g)} & , g \in G_{e_1} \\ t_{e_2}^{-1} \alpha_{e_2}(g) t_{e_2} = \omega_{e_2}(g) & , g \in G_{e_2} \\ \cancel{t_{e_1} = 1 = t_{\bar{e}_1}} & \end{array} \right\rangle$$

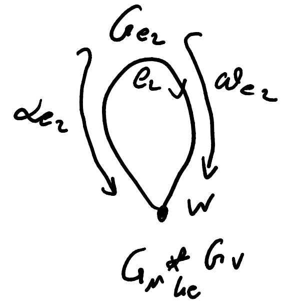
$G_u \ast_{G_e} G_v$

$$= \left\langle G_u \ast_{G_e} G_v, t_e, t_{\bar{e}} \;\middle|\; \begin{array}{l} t_{e_2} t_{\bar{e}_2} = 1 \\ t_{e_2}^{-1} \alpha_{e_2}(g) t_{e_2} = \omega_{e_2}(g), g \in G_{e_2} \end{array} \right\rangle$$

$$= \Big\langle \;\cdots\; G_e \;\cdots \;\Big|\; t_{e_2}^{-1}\,\alpha_{e_2}(g)\,t_{e_2} = \omega_{e_2}(g),\ g\in G_{e_2} \Big\rangle$$

$= \pi_1(G, Y', w)$, where

Y':



"collapse e_1 & amalgamate vertex groups"

Now analogy ex. 2

iii) If (G,Y) is an arbitrary gog and T is a spanning tree, one obtains $\pi_1(G,Y,T)$ from $\pi_1(G,T,T)$ through iterated applications of HNN extensions. One can obtain $\pi_1(G,T,T)$ from a segment of groups •——• by iterated application of amalg. prod.

(also works by first collapsing edges, then doing HNN extension as we saw in the exercise above)

Thm 8.4: Let (G,Y) be a graph of groups, $P\in Y^0$ and $T\subseteq Y$ a spanning tree. The canonical homomorphism $p: F(G,Y)\to\pi_1(G,Y,T)$ restricts to an isomorphism $\pi_1(G,Y,P)\to\pi_1(G,Y,T)$.

Pf.: For every $v\in Y^0\setminus\{P\}$, there is a unique red. path $e_1\cdots e_k$ in T from P to v. Set $\gamma_v := t_{e_1}\cdot t_{e_2}\cdots t_{e_k} \in F(G,Y)$ and $\gamma_P := 1$. Define a map

$$q: \pi_1(G,Y,T) \longrightarrow \pi_1(G,Y,P) \quad \text{via}$$

$$g \longmapsto \gamma_v\, g\, \gamma_v^{-1} \quad \text{if } g\in G_v$$

$$t_e \longmapsto \gamma_{\alpha(e)}\, t_e\, \gamma_{\omega(e)}^{-1}.$$

$$t_e \mapsto \gamma_{\alpha(e)}\, t_e\, \gamma_{\omega(e)}^{-1}.$$

This yields a hom. as

$$q(t_e t_{\bar e}) = \gamma_{\alpha(e)}\, t_e\, \gamma_{\omega(e)}^{-1}\, \gamma_{\alpha(\bar e)}\, t_{\bar e}\, \gamma_{\omega(\bar e)}^{-1} = 1$$

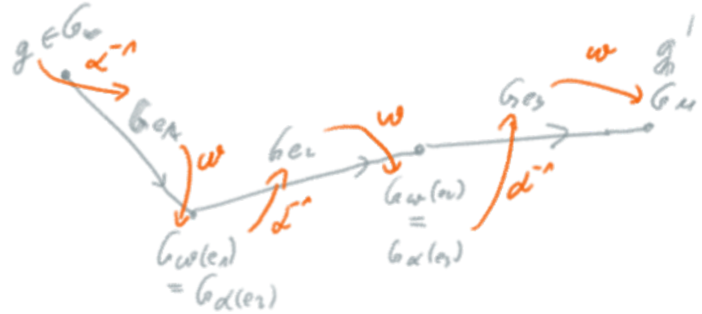
$$q(t_e^{-1}\alpha_e(g)\, t_e\, (\alpha_{\bar e}(g))^{-1}) = \dots = 1$$

One can check that $p \circ q = id$ and $q \circ p = id$ $\square$

Cor. 8.5: The fundamental groups $\pi_1(G, Y, P)$ and $\pi_1(G, Y, T)$ are isomorphic for any choice of vertex P and spanning tree T.

Def. 8.6: Let (G, Y) be a graph of groups and $T \subseteq Y$ a spanning tree.

- Let $g \in G_v$ and $g' \in G_u$, where $u, v \in T^0$. We say that <u>g and g' are equivalent</u> (w.r.t. T) if
  $$g' = \omega_{e_k}\alpha_{e_k}^{-1} \dots \omega_{e_1}\alpha_{e_1}^{-1}(g),$$
  where $e_1 \dots e_k$ is a path in T from v to u; we also consider g as equivalent to itself.



- Let Y_+^1 be an orientation of Y. Any element $x \in \pi_1(G, Y, T)$ can be written as $x = g_1 \dots g_n$, where $g_i \in G_v$ for some $v \in Y^0$ or $g_i = t_e^{\pm 1}$ for some $e \in Y_+^1 \setminus T^1$. Such an expression is called <u>reduced</u> if:

i) for no i, g_i and g_{i+1} are equivalent to elements of the same vertex group; (in part. not contained in same v group)

ii) there is no subword of the form $t_e t_e^{-1}$ or $t_e^{-1} t_e$;

iii) there is no subword of the form $t_e^{-1} \dots t_e$ where

ii) there is no subword of the form $t_e t_e^{-1}$ or $t_e^{-1} t_e$;

iii) there is no subword of the form $t_e^{-1} g t_e$, where g is equivalent to an element from $\alpha_e(G_e)$;

iv) there is no subword of the form $t_e g t_e^{-1}$, where g is equivalent to an element from $\omega_e(G_e)$.