Introduction to Non-Collisional Kinetic Theory

Dr. Ludovic Cesbron

Topics in Non-Collisional Kinetic Theory

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Outline

The Kinetic Theory

Kinetic Theory emanates from Statistical Mechanics, which is the study of mechanical system formed of a large number of elements. The core idea of Kinetic theory can be stated as follows:

Statement

The behaviour of a fluid can be entirely determined by the motion of the particles that constitute this fluid.
In other words, the evolution of any macroscopic observables (temperature, pressure, velocity, density...) can be derived from the underlying Newton dynamics which govern the motion of particles.

Historical Context

Fathers of Kinetic Theory:

- James Clerk Maxwell, 1831-1879
- Ludwig Boltzmann, 1844-1906
- (Rudolf Clausius, 1822-1888)

Alternative theory: Caloric theory (Lavoisier, Laplace, Carnot, Poisson...)

Microscopic scale

→ Newton dynamics

1 particle → $(X(t), V(t))$ position & velocity $\in \mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{cases} \dot{X}(t) = V(t) \\ \dot{V}(t) = a(t, X(t), V(t)) \end{cases} \quad \Big| \; 6 \text{ equations}$$

In a fluid, one can assume that there are $\sim 10^{23}$ particles

→ This scale is not relevant to describe a fluid

* Too many equations
* The information is not directly relevant
* Initial condition

Macroscopic scale

→ fluid dynamics : empirical sciences

Unknowns of the equations are macroscopic observables

Famous examples: Diffusion equations (heat equation)
Euler equations
Navier-Stokes equations

Assumption: fluids are modelled as continua

Mesoscopic scale

→ Kinetic equations

* "Fictive" scale fit to study phenomena which are not macroscopically observable and yet involve a large enough number of particles for Statistical Mechanics to apply.

* The unknown of kinetic equation is called a distribution function, it is a probability density.

Microscopic
Newton

N-particle limit

Mesoscopic
Kinetic

Macroscopic
fluid

Hydrodynamical limit

The Phase Space

The phase space is the space of all possible states occurring in the mathematical model of some physical system.

If we study a deterministic system described by an evolution equation, then the phase space is the "smallest" space on which the equation determines a unique, well-behaved solution.

In the context of this lecture, the phase space of a particle obeying Newton's laws is made of position and velocity (x, v).

Type A & type B $\rightarrow (A, x, v)$

The distribution function

$\iint f(t,x,v)\,dx\,dv = 1$

Consider a system of identical point particles. If the total number of particle per unit of volume is large enough then the state of the system at time t can be described statistically in the single particle phase space.

We consider the distribution function $f \equiv f(t, x, v)$ that is the number density of particles which are located at the position x and have the velocity v at time t. In other words, if Ω and $\mathcal{V}$ are subsets of $\mathbb{R}^3$ then the total number of particles $\mathcal{N}_{\Omega,\mathcal{V}}$ which have positions in Ω and velocity in $\mathcal{V}$ at time t is given by

$$\mathcal{N}_{\Omega,\mathcal{V}}(t) = \iint_{\Omega\times\mathcal{V}} f(t, x, v)\,\mathrm{d}x\,\mathrm{d}v.$$

Note that $f(t, \cdot, \cdot)$ is in fact a probability density.

The distribution function

More generally, given a (additive) physical quantity for a particle such as momentum or energy, we can express the corresponding quantity for the portion of the particle system with position in Ω by integrals of f.

- Momentum: The momentum of a particle with mass m and velocity v is given by $\phi(v) = mv$ so the total momentum of particles with position in Ω is

$$P_\Omega(t) = \iint_{\Omega \times \mathbb{R}^3} mv f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

- Energy: The kinetic energy of that same particle is given by $\phi(v) = \frac{1}{2}m|v|^2$, hence the total kinetic energy of particles in Ω is given by

$$\mathcal{E}_\Omega(t) = \iint_{\Omega \times \mathbb{R}^3} \frac{1}{2}m|v|^2 f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Note that P_Ω and $\mathcal{E}_\Omega$ are macroscopic observables!

Key questions in Kinetic Theory

In this class, we will tackle some of the most fundamental questions of Kinetic Theory in the context of non-collisional plasma physics:

- Rigorous derivation of Kinetic equations from Newton dynamics: How do you derive a PDE on the distribution function f from the system of ODEs describing the motion of particles $(X(t), V(t))$?
- Well-posedness of the kinetic equation
- Qualitative analysis, stability, long-time behaviour of a solution to the kinetic equation.

Plasmas

Plasma is state of matter. A plasma is an ionised gas, i.e. a gas which contains a significant amount of ions and free electrons. Plasmas can be artificially generated by heating a neutral gas, or by embedding a gas into a strong magnetic field to the point where the gas becomes increasingly electrically conductive. Exemples of plasmas: stars, interstellar/intergalactic medium, solar wind, lightning...

Formal derivation of a Vlasov equation

Newton dynamics : $\begin{cases} \dot{X}(t) = V(t) & , X(t_0) = x_0 \\ \dot{V}(t) = a(t, X(t), V(t)) & , V(t_0) = v_0 \end{cases}$ (N)

$t_0 \geq 0$, $(x_0, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$

A solution of (N) is $(X(t; t_0, x_0, v_0), V(t; t_0, x_0, v_0))$.

Consider $\Theta_{t_0} \subset \mathbb{R}^3 \times \mathbb{R}^3$ and define Θ_t as

$$\Theta_t = \left\{ (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \text{ s.t. } \begin{cases} x = X(t; t_0, x_0, v_0) \\ v = V(t; t_0, x_0, v_0) \end{cases} \text{ for } (x_0, v_0) \in \Theta_{t_0} \right\}$$

$$\iint_{\Theta_{t_0}} f(t_0, x_0, v_0)\, dx_0\, dv_0 = \iint_{\Theta_t} f(t, x, v)\, dx\, dv$$

$$= \iint_{\Theta_t} f(t, X(t; t_0, x_0, v_0), V(t; t_0, x_0, v_0))\, dX_t\, dV_t$$

Formal derivation of a Vlasov equation

Assumption : the substitution $(x_0, v_0) \longmapsto (X(t; t_0, x_0, v_0), V(t; t_0, x_0, v_0))$

is Lebesgue measure preserving $dx_0 dv_0 = dX_t dV_t$

↳ reasonable if we neglect collisions

For any Θ_{t_0}

$$\iiint_{\Theta_{t_0}} f(t_0, x_0, v_0)\, dx_0 dv_0 = \iint_{\Theta_{t_0}} f(t, X(t; t_0, x_0, v_0), V(t; t_0, x_0, v_0))\, dx_0 dv_0$$

$$\Rightarrow \quad f(t_0, x_0, v_0) = f(t, X(t; t_0, x_0, v_0), V(t; t_0, x_0, v_0))$$

$\frac{d}{dt}$:

$$\boxed{0 = \partial_t f + v \cdot \nabla_x f + a(t, x, v) \cdot \nabla_v f}$$

Vlasov Equation

Interaction Potentials

Newton's law : $m\ddot{X}_i(t) = \sum_j F_{j\to i}(t)$

$F_{j\to i}$ is the force exerted by the jth particle onto the ith part.

If the interaction is translation invariant then "often" the force derives from an interaction potential $W : \mathbb{R}^d \to \mathbb{R}$

$$F_{j\to i} = -\nabla W(x_i - x_j) \quad , \quad c>0$$

<u>Exples</u> :

$$* \; W(x_i - x_j) = c\,\frac{q_i q_j}{|x_i - x_j|^{d-2}}$$

electrical charge of particle i

Electrostatic Potential

Coulomb's force

$$* \; W(x_i - x_j) = -c\,\frac{m_i m_j}{|x_i - x_j|^{d-2}}$$

mass

gravitational potential

→ Newton force

Interaction Potentials

As $N \to +\infty$

$$m\ddot{x}(t) = \int_{\mathbb{R}^3} F_{y\to x(t)}\, \rho(t,y)\, dy$$

with $\rho(t,y) = \int_{\mathbb{R}^3} f(t,y,v)\, dv$ is the macroscopic density.

$$\Rightarrow m\ddot{x}(t) = \int_{\mathbb{R}^3} -\nabla W(y - x(t))\, \rho(t,y)\, dy = -\nabla(W * \rho_t)(x(t))$$

Vlasov Equation becomes

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = 0 \\ F = -\nabla(W * \rho_t)(y), \quad \rho_t(y) = \int_{\mathbb{R}^3} f(t,y,v)\, dv \end{cases}$$

The Vlasov-Poisson model

Reminder: if $W(x) = \frac{\pm c}{|x|^{d-2}}$, $\begin{matrix} d \geq 3 \\ c > 0 \end{matrix}$ then W is the fundamental solution of the Laplace equation: $\pm \Delta W = \delta_0$

Hence: $U = W * \rho \Rightarrow \Delta U = \rho$

Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0, \\ F = -\nabla U, \quad \pm \Delta U = \rho_t, \\ \rho_t(y) = \int f(t, y, v)\, dv, \\ f(0, x, v) = f_{in}(x, v) \end{cases}$$

+ gravitational

− electrostatic

Energy of the system

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + F\cdot\nabla_v f = 0 \\ F = -\nabla U, \quad \pm\Delta U = \rho \end{cases}$$

Definitions:

- $\displaystyle Kin(t) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 f(t,x,v)\,dx\,dv$ Kinetic energy

- $\displaystyle Pot(t) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} U(t,x) f(t,x,v)\,dx\,dv$ Potential energy

$$= \int_{\mathbb{R}^3} (W * \rho_t)(x)\,\rho(t,x)\,dx$$

- Total Energy: $\mathcal{E}_t = Kin(t) + \frac{1}{2}Pot(t)$

Conservation of total Energy

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0 \\ F = -\nabla U, \quad \pm \Delta U = \rho \end{cases}$$

$$\frac{d}{dt} \mathcal{E}_t = \iint \frac{1}{2} |v|^2 \partial_t f \, dx dv + \frac{1}{2} \int (W * \partial_t \rho) \rho \, dx + \frac{1}{2} \int (W * \rho) \partial_t \rho \, dx$$

$$= \iint \frac{1}{2} |v|^2 \partial_t f \, dx dv + \int (W * \rho) \partial_t \rho \, dx$$

Continuity equation : $\int dv$ the Vlasov equation

$$\boxed{\partial_t \rho + \nabla_x \cdot \int v f \, dv = 0}$$

Conservation of total Energy

$$\frac{d}{dt}\mathcal{E}_r = \iint \frac{1}{2}|v|^2 \Big(\underbrace{-v\cdot\nabla_x f}_{=0} \underbrace{- F\cdot\nabla_v f}\Big)\,dx\,dv - \iint (W*\rho)\,\nabla_x(vf)\,dx\,dv$$

$$= \iint (v\cdot F)\, f\,dx\,dv + \iint \underbrace{\nabla_x(W*\rho)}_{=-F}\cdot v f\,dx\,dv$$

$$= 0$$

$\rightarrow$ Conservation of total energy.

Conservation of total Energy in the Coulombian case

$$\mathrm{Pot}(t) = \int u(t,x)\,\rho(t,x)\,dx = -\int u \Delta u \,dx$$

$$= \int |\nabla u|^2 dx \geq 0 \quad \int F^2 dx$$

$$\mathcal{E}_t = \mathrm{Kin}(t) + \mathrm{Pot}(t) = \text{constant}$$

$$\Rightarrow \begin{cases} \mathrm{Kin}(t) \leq C & \leftarrow \text{the } 2^{nd} \text{ moment of } f \text{ is bounded} \\ \mathrm{Pot}(t) \leq C & \leftarrow F \in L^2(\mathbb{R}^d) \end{cases}$$

General Conservations

$\beta : \mathbb{R} \to \mathbb{R}$, $\beta \in \mathcal{C}^1(\mathbb{R})$

Then $\boxed{\dfrac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f) \, dx\,dv = 0}$

proof:

$$\partial_t f \, \beta'(f) + v \cdot \nabla_x f \, \beta'(f) + F \cdot \nabla_v f \, \beta'(f) = 0$$

$$\partial_t (\beta(f)) + v \cdot \nabla_x (\beta(f)) + F \cdot \nabla_v (\beta(f)) = 0$$

$\iint dx\,dv$

$$\Rightarrow \quad \frac{d}{dt} \iint \beta(f) \, dx\,dv + 0 + 0 = 0 \qquad \text{qed}$$

General Conservations

* $\beta(s) = |s|^p$, $p \geq 1$ L^p-norms

$$\longrightarrow \iint |f(t,x,v)|^p \, dx\,dv = \iint |f_{in}(x,v)|^p \, dx\,dv \quad \underline{\forall t > 0}$$

* $\beta(s) = s\log(s)$ Entropy

$$\longrightarrow \iint f \log(f) \, dx\,dv = \iint f_{in} \log(f_{in}) \, dx\,dv \quad \underline{\forall t > 0}$$

$\Longrightarrow$ The Vlasov-Poisson system conserves entropy