PROPAGATION OF MOMENTS IN DIMENSION 2 AND PROPAGATION OF C^1 REGULARITY IN DIMENSIONS 2 AND 3

CAMILLA NAIARETTI

Introduction

This text has two main goals: the first is to analyse the propagation of moments in dimension 2 and the second is to analyse the propagation of C^1 regularity in dimensions 2 and 3.
The propagation of moments in dimension 2 represents the key step in the proof of global existence of classical solutions of the Vlasov-Poisson system. It is also central in studying the propagation of C^1 regularity in dimensions 2 and 3.
In the part of this work about the propagation of C^1 regularity in dimensions 2 and 3 we will in particular prove the first step, namely an L^∞ bound on the field, of the proof of C^1 regularity of the weak solution (f, E) of the Cauchy problem for Vlasov-Poisson system.

1. Propagation of Moments in Dimension 2

The aim of this chapter is to bound moments in the variable v of solutions of the Vlasov-Poisson system. In particular, we want to construct weak solutions $f \equiv f(t, x, v)$ of the Vlasov-Poisson systems for which quantities

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k f(t,x,v)\,dxdv \tag{1}$$

are bounded for all $t \geq 0$, provided that they are bounded at $t = 0$. This estimations are central in the proof of the uniqueness part of the global existence of the classical (instead of weak) solutions of the Vlasov-Poisson system and for the proof of the C^1 regularity of section 4.5 of [1].
We consider the case $d = 2$, which is the easiest one.

Theorem 1. *Let $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ be such that $f^{in} \geq 0$ a.e., and assume that*

$$\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f^{in}(x,v)\,dxdv + \frac{1}{2}\int_{\mathbb{R}^2} |E^{in}(x)|^2 dx =: \mathcal{E}^{in} < \infty,$$

where

$$E^{in} = -\nabla G_2 * \rho^{in}, \quad \rho^{in} := \int_{\mathbb{R}^2} f^{in} dv.$$

Assume further that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^{k_0} f^{in}(x,v)\,dxdv < \infty,$$

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for some $k_0 > 2$.
Then, there exists a weak solution of the Vlasov-Poisson system in $\mathbb{R}^2 \times \mathbb{R}^2$ *with initial data* f^{in} *such that*

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k f(t,x,v)dxdv \leq e^{C_k\left(\mathcal{M}^{in}, \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\right)t} \iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k f^{in}(x,v)dxdv$$

for all $t \geq 0$ *and* $0 \leq k \leq k_0$.

Remark 1. Two remarks are in order:

- Strictly speaking, the proof of Theorem 1 does not directly exploit the assumption
$$\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f^{in}(x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^2} |E^{in}(x)|^2 dx =: \mathcal{E}^{in} < \infty.$$
Nonetheless, this assumption is needed in Arsenev theorem, which provides the existence of a weak solution of the Vlasov-Poisson system, and to prove the theorem about propagation of moments in dimension 3 (see section 4.4 of [1]).
- Theorem 1 gives a bound of the quantity (1), that is
$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k f(t,x,v)dxdv,$$
providing that at the initial time $t = 0$
$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^{k_0} f^{in}(x,v)dxdv < \infty \tag{2}$$
for some $k_0 > 2$. Therefore, we bound the desired quantity (1) by
$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k f^{in}(x,v)dxdv$$
for $0 \leq k \leq k_0$. To note that the bound of (1) actually hold for all $k \geq 0$ but when $k > k_0$ quantity (2) can be unbounded, hence the inequality in Theorem 1 is trivial.

Proof. We start from the differential inequality satisfied by moments of the distribution function:

$$\begin{aligned}\frac{\partial}{\partial t}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k f(t,x,v)dxdv &= k\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^{k-2} v\cdot E(t,x) f(t,x,v)dxdv \\ &\leq k\int_{\mathbb{R}^2} |E(t,x)| \left(\int_{\mathbb{R}^2} |v|^{k-1} f(t,x,v)dv\right)dx,\end{aligned}$$

where the right hand side of the inequality follows from Tonelli (since all quantities are positive). The first equality comes from the definition of the Vlasov-Poisson equation and from an integration by parts. Recall that the Vlasov-Poisson equation is

$$\partial_t f + v\cdot\nabla_x f + E(t,x)\cdot\nabla_v f = 0$$

and the divergence form of it is

$$\partial_t f + \operatorname{div}_x(vf) + \operatorname{div}_v(E(t,x)f) = 0.$$

From the divergence form of the Poisson-Vlasov equation follows that

$$\partial_t f = -\operatorname{div}_x(vf) - \operatorname{div}_v(E(t,x)f).$$

Therefore, we obtain

$$\frac{\partial}{\partial t}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k f(t,x,v)dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k\partial_t f(t,x,v)dxdv$$

$$= -\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k \operatorname{div}_x(vf(t,x,v))dxdv - \iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k \operatorname{div}_v(E(t,x)f(t,x,v))dxdv$$

$$= -\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k v\cdot\nabla_x(f(t,x,v))dxdv - \iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k E(t,x)\cdot\nabla_v(f(t,x,v))dxdv.$$

At this point, we integrate both integrals by parts. We work firstly on the integral

$$\int_{\mathbb{R}^2}|v|^k v\cdot\nabla_x(f(t,x,v))dx.$$

Computing the derivative with respect to x of $|v|^k v$ we get $\nabla_x(|v|^k v)=0$, and integrating $\nabla_x(f(t,x,v))$ we obtain f. Since $f(t,x,v)\to 0$ as $x\to\infty$, the boundary term goes 0. Hence, we obtain

$$\int_{\mathbb{R}^2}|v|^k v\cdot\nabla_x(f(t,x,v))dx = 0 - \int_{\mathbb{R}^2}\nabla_x(|v|^k v)f(t,x,v)dx = -\int_{\mathbb{R}^2}0dx = 0.$$

Now, we work on the integral

$$\int_{\mathbb{R}^2}|v|^k E(t,x)\cdot\nabla_v(f(t,x,v))dv.$$

Computing the derivative with respect to v of $|v|^k$ we get $\nabla_v(|v|^k) = k|v|^{k-1}\frac{v}{|v|} = k|v|^{k-2}v$, and integrating $E(t,x)\cdot\nabla_v(f(t,x,v))$ we obtain $E(t,x)f$. Since $f(t,x,v)\to 0$ as $v\to\infty$, the boundary term goes 0. Hence, we obtain

$$\int_{\mathbb{R}^2}|v|^k E(t,x)\cdot\nabla_v(f(t,x,v))dv = 0 - \int_{\mathbb{R}^2}k|v|^{k-2}v\cdot E(t,x)f(t,x,v)dv.$$

So, we finally obtain

$$-\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k v\cdot\nabla_x(f(t,x,v))dxdv - \iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k E(t,x)\cdot\nabla_v(f(t,x,v))dxdv$$

$$= 0 - \left(-\iint_{\mathbb{R}^2\times\mathbb{R}^2}k|v|^{k-2}v\cdot E(t,x)f(t,x,v)dxdv\right)$$

$$= k\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^{k-2}v\cdot E(t,x)f(t,x,v)dxdv,$$

which is the desired result. Then, Hödler inequality ($\frac{1}{k+2}+\frac{k+1}{k+2}=1$) gives

$$\frac{\partial}{\partial t}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k f(t,x,v)dxdv \le k\int_{\mathbb{R}^2}|E(t,x)|\left(\int_{\mathbb{R}^2}|v|^{k-1}f(t,x,v)dv\right)dx$$

$$\le k\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)}\left\|\int_{\mathbb{R}^2}|v|^{k-1}f(t,\cdot,v)dv\right\|_{L^{\frac{k+2}{k+1}}(\mathbb{R}^2)}.$$

Recall that the interpolation inequality states that for each $m\ge r>0$, there exists a positive constant $C(d,m,r)$ such that, for each measurable function $f\equiv f(x,v)$ defined a.e. on $\mathbb{R}^d_x\times\mathbb{R}^d_v$, one has

$$\left\|\int_{\mathbb{R}^2}|v|^r f(\cdot,v)dv\right\|_{L^{\frac{m+d}{r+d}}(\mathbb{R}^d)} \le C(d,m,r)\|f\|^{\frac{m-r}{m+d}}_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\||v|^m f\|^{\frac{r+d}{m+d}}_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Using it with $r = k-1$, $d = 2$, and $m = k$ we get

$$\begin{aligned}\left\|\int_{\mathbb{R}^2}|v|^{k-1}f(t,\cdot,v)dv\right\|_{L^{\frac{k+2}{k+1}}(\mathbb{R}^2)} &\leq C\|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{k-(k-1)}{k+2}}\||v|^k f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{k-1+2}{k+2}} \\ &\leq C\|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{1}{k+2}}\||v|^k f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{k+1}{k+2}} \\ &\leq C_1\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k f(t,x,v)dxdv\right)^{\frac{k+1}{k+2}},\end{aligned}$$

where the last inequality follows from the conservation of the L^p norm and from the assumption that $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. We recall that the conservation of the L^p norm states that for all $p \in [1,\infty]$

$$\|f\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} = \|f^{in}\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}.$$

In particular, in this case we have that

$$\|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{1}{k+2}} = \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{1}{k+2}} < \infty,$$

where the last inequality follows from the assumption $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, which implies that $f^{in} \in L^\infty$, i.e., the L^∞ norm of f^{in} is bounded. Denoting

$$\mu_k(t) := \iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k f(t,x,v)dxdv,$$

one has

$$\begin{aligned}\dot{\mu}_k(t) &= \frac{\partial}{\partial t}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k f(t,x,v)dxdv \\ &\leq k\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)}\left\|\int_{\mathbb{R}^2}|v|^{k-1}f(t,\cdot,v)dv\right\|_{L^{\frac{k+2}{k+1}}(\mathbb{R}^2)} \\ &\leq k\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)}C_1\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^k f(t,x,v)dxdv\right)^{\frac{k+1}{k+2}} \\ &\leq C_2\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)}\mu_k(t)^{\frac{k+1}{k+2}}.\end{aligned}$$

Our goal is to obtain an inequality of the form

$$\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)} \leq C\mu_k(t)^{\frac{1}{k+2}}.$$

Indeed, such an equality directly gives

$$\dot{\mu}_k(t) \leq C_2\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)}\mu_k(t)^{\frac{k+1}{k+2}} \leq C_2C\mu_k(t)^{\frac{1}{k+2}}\mu_k(t)^{\frac{k+1}{k+2}} \leq C_3\mu_k(t),$$

which, by Grönwall Lemma, allows to conclude

$$\mu_k(t) \leq \mu_k(0)e^{\int_0^t C_3 ds} \leq \mu_k(0)e^{C_3 t},$$

which is exactly the desired result.

From [1, page 55], an a priori bound on the force field reads

$$\|E(t,\cdot)\|_{L^q} \leq C\|\rho(t,\cdot)\|_{L^p} \quad \forall\, t \geq 0,$$

with

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{d-1}{d} \quad \text{if} \quad 1 < p < d.$$

Choosing $q = k+2$ and $p = \frac{2k+4}{k+4}$ we have that $1 < p = \frac{2k+4}{k+4} < 2 = d$ and $1 + \frac{1}{q} = \frac{1}{p} + \frac{d-1}{d}$. Hence, an a priori bound on the force field is

$$\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)} \leq \tilde{C}\|\rho(t,\cdot)\|_{L^{\frac{2k+4}{k+4}}(\mathbb{R}^2)}.$$

Applying the interpolation inequality with $m = k$, $d = 2$, and $r = 0$ we obtain

$$\begin{aligned}
\|\rho(t,\cdot)\|_{L^{\frac{k+2}{2}}(\mathbb{R}^2)} &= \left\| \int_{\mathbb{R}^2} f(\cdot, v) dv \right\|_{L^{\frac{k+2}{2}}(\mathbb{R}^2)} \\
&= \left\| \int_{\mathbb{R}^2} |v|^0 f(\cdot, v) dv \right\|_{L^{\frac{k+2}{2}}(\mathbb{R}^2)} \\
&\leq C\|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{k}{k+2}} \||v|^k f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{2}{k+2}} \\
&\leq C_4 \left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k f(t,x,v) dx dv \right)^{\frac{2}{k+2}} \\
&= C_4 \mu_k(t)^{\frac{2}{k+2}},
\end{aligned}$$

where we use that $\|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{k}{k+2}} < \infty$ (L^p norm is conserved for every $p \in [1,\infty]$ and by assumption $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$) and the definition of μ_k.

The Interpolation proposition states that under the assumption $1 \leq p \leq r \leq q \leq \infty$ and $\theta \in (0,1)$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$, if $f \in L^p(\Omega) \cap L^q(\Omega)$, then $f \in L^r(\Omega)$ and

$$\|f\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)}^\theta \|f\|_{L^q(\Omega)}^{1-\theta}.$$

Hence, we obtain

$$\|\rho(t,\cdot)\|_{L^{\frac{2k+4}{k+4}}(\mathbb{R}^2)} \leq \|\rho(t,\cdot)\|_{L^1(\mathbb{R}^2)}^{1-\theta} \|\rho(t,\cdot)\|_{L^{\frac{k+2}{2}}(\mathbb{R}^2)}^\theta,$$

if we choose $\theta = \frac{1}{2}$. Indeed,

$$\frac{k+4}{2k+4} = \frac{\frac{1}{2}}{\frac{k+2}{2}} + \frac{1-\frac{1}{2}}{1}.$$

Since

$$\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f(t,x,v) dv \right) dx,$$

$f^{in} \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and the L^1 norm is conserved, we have that

$$\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} < \infty.$$

Overall, we obtain

$$\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)} \leq \tilde{C}\|\rho(t,\cdot)\|_{L^{\frac{2k+4}{k+4}}(\mathbb{R}^2)} \leq C_5 \|\rho(t,\cdot)\|_{L^{\frac{k+2}{2}}(\mathbb{R}^2)}^{\frac{1}{2}} \leq C_6 \mu_k(t)^{\frac{1}{k+2}},$$

where the last inequality follows from the bound $\|\rho(t,\cdot)\|_{L^{\frac{k+2}{2}}(\mathbb{R}^2)} \leq C_4 \mu_k(t)^{\frac{2}{k+2}}$. Inserting the above bound for the norm of the force field in the differential inequality of μ_k, we find

$$\begin{aligned}\dot{\mu}_k(t) &\leq C_2\|E(t,\cdot)\|_{L^{k+2}(\mathbb{R}^2)}\mu_k(t)^{\frac{k+1}{k+2}}\\ &\leq C_2C_6\mu_k(t)^{\frac{1}{k+2}}\mu_k(t)^{\frac{k+1}{k+2}}\\ &\leq C_k(\mathcal{M}^{in},\|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)})\mu_k(t).\end{aligned}$$

Then, by Grönwall Lemma, we conclude that

$$\mu_k(t) \leq \mu_k(0)e^{C_k\left(\mathcal{M}^{in},\|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\right)t},$$

for all $t, k \geq 0$, which is the desired result. □

2. Propagation of C^1 Regularity in Dimensions 2 and 3

In this section, we seek to understand how propagation of moments can be used to establish propagation (for all positive times) of C^1 regularity of the initial data. Simply said, we want to prove that if the initial condition f^{in} of the Cauchy problem for the Vlasov-Poisson system is C^1 then the weak solution of the problem is also C^1.

In statistical mechanics compactly supported densities are not very natural. For instance, Maxwell-Boltzmann distributions are not compactly supported in v, but enjoy excellent decay properties as $|v| \to \infty$. The decay of the distribution function f in the v variable can be reformulated in terms of a convenient weighted estimate: the decay of the distribution function f can be expressed by a class of weight functions w. In particular, let $w \in C^1(\mathbb{R})$ be such that

$$w \geq 0, \quad w' \leq 0, \quad \text{and} \quad w(r) = O(r^{-\alpha}) \quad \text{with} \quad \alpha > d.$$

We formalize our discussion in the following theorem.

Theorem 2. *Assume that*

$$0 \leq f^{in}(x,v) \leq w(|v|) \quad \text{and} \quad \nabla G_d * \rho^{in} \in L^2(\mathbb{R}^d),$$

where we have denoted

$$\rho^{in} := \int_{\mathbb{R}^d} f^{in} dv.$$

In addition, assume that, for some k_0, one has

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}(1+|v|^{k_0})f^{in}(x,v)dxdv < \infty,$$

with

$$k_0 > d(d-1),$$

i.e.

$$k_0 > 2 \quad \text{if} \quad d=2, \quad \text{while} \quad k_0 > 6 \quad \text{if} \quad d=3.$$

Then there exists a weak solution $(f,E) \in C^1(\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d)\times C^1(\mathbb{R}_+\times\mathbb{R}^d)$ of the Cauchy problem for the Vlasov-Poisson system satisfying the initial condition $f|_{t=0} = f^{in}$, together with the decay estimates

$$f(t,x,v) + |\mathrm{D}_x f(t,x,v)| + |\mathrm{D}_v f(t,x,v)| = O(|v|^{-\alpha}) \quad \text{as} \quad |v| \to \infty$$

uniformly in $(t,x) \in [0,T]\times\mathbb{R}^d$.

Remark 2. The theorem above does not stated the uniqueness of the weak solution f of the Cauchy problem for the Vlasov-Poisson system. The uniqueness can be obtained as a consequence of some estimates used for the propagation of regularity.

The proof is divided into 6 steps. We present the first one: an L^∞ bound on the field.

Proof of Step 1: L^∞ bound on the field. By Theorem 1 and Theorem 4.4.1 of [1] we have

$$\mu_k(t) \leq C_T$$

for all $k = 0, \ldots, k_0$ and all $t \in [0, T]$. Indeed, we proved that

$$\mu_k(t) \leq e^{C_k t}\mu_k(0) \quad \text{for all} \quad t \geq 0,$$

and therefore for all $t \in [0, T]$

$$\mu_k(t) \leq \max_{t\in[0,T]} e^{C_k t}\mu_k(0) = C_T,$$

since $\mu_k(0) < \infty$ for $0 \leq k \leq k_0$. Then, the interpolation inequality gives

$$\|\rho(t, \cdot)\|_{L^{\frac{k_0+d}{d}}(\mathbb{R}^d)} \leq C_T$$

for all $t \in [0, T]$.
By conservation of mass [1, page 53], we get, for all $t \geq 0$,

$$\|\rho(t, \cdot)\|_{L^1(\mathbb{R}^d)} = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t, x, v)dvdx = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t, x, v)dxdv \leq \mathcal{M}^{in},$$

where we use Tonelli to exchange the order of the integrals (since f is positive).
Since the force field is given by

$$E(t, \cdot) = -\nabla G_d * \rho(t, \cdot),$$

we get that

$$E(t, \cdot) = -(\mathbb{1}_{B(0,1)}\nabla G_d) * \rho(t, \cdot) - (\mathbb{1}_{B(0,1)^c}\nabla G_d) * \rho(t, \cdot).$$

In addition we know that

$$\mathbb{1}_{B(0,1)}\nabla G_d = O(|x|^{1-d}\mathbb{1}_{|x|\leq1}) \in L^m(\mathbb{R}^d) \quad \text{for all} \quad 1 < m < \frac{d}{d-1}$$

and

$$\mathbb{1}_{B(0,1)^c}\nabla G_d = O(|x|^{1-d}\mathbb{1}_{|x|\geq1}) \in L^\infty(\mathbb{R}^d).$$

Young's convolution inequality states that for $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, and r such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $1 \leq p, q, r \leq \infty$, we have

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

Since $\frac{k_0+d}{d} > d$, we obtain

$$\|(\mathbb{1}_{B(0,1)}\nabla G_d) * \rho(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq C_d\|\rho(t, \cdot)\|_{L^{\frac{k_0+d}{d}}(\mathbb{R}^d)},$$

and

$$\|(\mathbb{1}_{B(0,1)^c}\nabla G_d) * \rho(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq C'_d\|\rho(t, \cdot)\|_{L^1(\mathbb{R}^d)}.$$

This implies that

$$\begin{aligned}\|E(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} &\leq \|(\mathbb{1}_{B(0,1)}\nabla G_d) * \rho(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} + \|(\mathbb{1}_{B(0,1)^c}\nabla G_d) * \rho(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \\ &\leq C_d \|\rho(t,\cdot)\|_{L^{\frac{k_0+d}{d}}(\mathbb{R}^d)} + C'_d \|\rho(t,\cdot)\|_{L^1(\mathbb{R}^d)} \\ &\leq C_d C_T + C'_d \mathcal{M}^{in} =: A_T.\end{aligned}$$

Therefore, we proved that the L^∞-norm of the force field is bounded. □

References

[1] F.Golse. *Mean Field Kinetic Equations*. 2013. Chap. IV, §3,5.