PROPAGATION OF C^1 REGULARITY FOR $d = 2, 3$

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We continue with the proof of Theorem 4.5.1:

Theorem (Theorem 4.5.1)**.** *Let $w \in C^1(\mathbb{R})$ such that*

$$w \geq 0, w' \leq 0, w(r) = O(r^{-\alpha}) \text{ with } \alpha > d.$$

Assume that $f^{in} \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$, $0 \leq f^{in}(x, v) + |D_x f^{in}(x, v)| + |D_v f^{in}(x, v)| \leq w(|v|)$, and $\nabla^2 G_d \star \rho^{in} \in L^2(\mathbb{R}^d)$, where $\rho^{in}(x) = \int_{\mathbb{R}^d} f^{in}(x, v)\mathrm{d}v$.
Assume that for some $k_0 > d(d-1)$ we have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (1 + |v|^{k_0}) f^{in}(x, v)\mathrm{d}x\mathrm{d}v < \infty.$$

Then there exists a weak solution $(f, E) \in C^1(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d) \times C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ for the Vlasov-Poisson equation satisfying the initial condition $f|_{t=0} = f^{in}$, together with the decay estimates

$$f(t, x, v) + |D_x f(t, x, v)| + |D_v f(t, x, v)| = O(|v|^{-\alpha}) \text{ as } |v| \to \infty$$

uniformly in $(t, x) \in [0, T] \times \mathbb{R}^d$.

Proof. We shall proceed with the proof in six steps.

Step 1. L^∞ bound on the field:
Recall that we have shown $\exists A = A_T$ such that for all $t \in [0, T]$ $\|E(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq A$.

Step 2. L^∞ bound on the charge density:
We show that for a.e. $(t, x, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$

$$f(t, x, v) \leq w(|v| - At).$$

Consider the quantity

$$h(t, x, v) := (f(t, x, v) - w(|v| - At))_+.$$

Then h is a weak solution to the following Vlasov equation with source term:

$$(\partial_t + v \cdot \nabla_x + E(t, x) \cdot \nabla_v) h(t, x, v) = (A - E(t, x) \cdot \frac{v}{|v|}) w'(|v| - At) \mathbb{1}_{f(t,x,v) \geq w(|v|-At)}.$$

We therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d \times \mathbb{R}^d} h(t, x, v)\mathrm{d}x\mathrm{d}v = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (A - E(t, x) \cdot \frac{v}{|v|}) w'(|v| - At) \mathbb{1}_{f(t,x,v) \geq w(|v|-At)} \mathrm{d}x\mathrm{d}v$$
$$\leq 0$$

since $w' \leq 0$ and $A - E(t, x) \cdot \frac{v}{|v|} \geq A - |E(t, x)| \geq 0$ a.e. $x \in \mathbb{R}^d$.
But by assumption, $h(0, x, v) = 0$ a.e. and $h \geq 0$ a priori, so

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} h(t, x, v)\mathrm{d}x\mathrm{d}v = 0$$

for all $t \in [0, T]$, hence $h(t, x, v) = 0$ and $f(t, x, v) \leq w(|v| - At)$ a.e.

We therefore have that for a.e. (t,x)

$$\begin{aligned}\rho(t,x) &= \int_{\mathbb{R}^d} f(t,x,v)\mathrm{d}v \le \int_{\mathbb{R}^d} w(|v|-At)\mathrm{d}v\\ &= \int_{B_{AT}(0)} w(|v|-At)\mathrm{d}v + \int_{\mathbb{R}^d\setminus B_{AT}(0)} w(|v|-At)\mathrm{d}v\\ &\le \int_{B_{AT}(0)} w(-AT)\mathrm{d}v + \int_{\mathbb{R}^d\setminus B_{AT}(0)} w(|v|-AT)\mathrm{d}v \quad \text{since } w \text{ is non-increasing}\\ &= w(-AT)(AT)^d|\mathbb{B}^d| + |\mathbb{S}^{d-1}|\int_{AT}^\infty w(r-AT)r^{d-1}\mathrm{d}r\\ &= w(-AT)(AT)^d|\mathbb{B}^d| + |\mathbb{S}^{d-1}|\int_0^\infty w(r)(r+AT)^{d-1}\mathrm{d}r =: R_T < \infty\end{aligned}$$

where the integral on the last line is finite because $w = O(r^{-\alpha})$ and $\alpha > d$. Therefore $\|\rho\|_{L^\infty} \le R_T$.

Step 3. L^∞ estimate for $D_{x,v}f$ in terms of D_xE:
Set

$$L(t) := \|D_xf(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} + \|D_vf(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.$$

We can differentiate the Vlasov equation in the weak sense to obtain

$$(\partial_t + v\cdot\nabla_x + E(t,x)\cdot\nabla_v)\begin{pmatrix}D_xf\\ D_vf\end{pmatrix} = -\begin{pmatrix}0 & D_xE^\top\\ I_d & 0\end{pmatrix}\begin{pmatrix}D_xf\\ D_vf\end{pmatrix},$$

so that a.e.

$$(\partial_t + v\cdot\nabla_x + E(t,x)\cdot\nabla_v)(|D_xf|+|D_vf|) \le (1+|D_xE|)(|D_xf|+|D_vf|).$$

Then setting

$$J(t) := \int_0^t (1+\|D_xE(s,\cdot)\|_{L^\infty(\mathbb{R}^d)})\mathrm{d}s$$

and

$$g(t,x,v) := e^{-J(t)}(|D_xf(t,x,v)|+|D_vf(t,x,v)|),$$

we have

$$\begin{aligned}(\partial_t + v\cdot\nabla_x + E\cdot\nabla_v)g &= e^{-J(t)}(\partial_t + v\cdot\nabla_x + E(t,x)\cdot\nabla_v)(|D_xf|+|D_vf|)\\ &\quad - J'(t)e^{-J(t)}(|D_xf|+|D_vf|)\\ &\le e^{-J(t)}(1+|D_xE|)(|D_xf|+|D_vf|)\\ &\quad - (1+\|D_xE(s,\cdot)\|_{L^\infty(\mathbb{R}^d)})e^{-J(t)}(|D_xf|+|D_vf|)\\ &= (|D_xE| - \|D_xE(s,\cdot)\|_{L^\infty(\mathbb{R}^d)})g \le 0 \text{ a.e.}\end{aligned}$$

Hence g satisfies a Vlasov equation with negative source term, and by the maximum principle,

$$g(t,x,v) \le \operatorname*{ess\ sup}_{x,v\in\mathbb{R}^d} g(0,x,v) \le L(0) \text{ a.e.}$$

thus

$$|D_{x,v}f| \le g(t,x,v)e^{J(t)} \le L(0)e^{J(t)} \text{ a.e.}$$

and

$$L(t) = \operatorname{ess}\sup_{x,v\in\mathbb{R}^d} |D_x f| + \operatorname{ess}\sup_{x,v\in\mathbb{R}^d} |D_x f| \leq 2L(0)e^{J(t)}.$$

Step 4. L^∞ estimate for $D_x\rho$ in terms of D_xE:
Proceeding similarly to Step 2, we have that

$$\begin{aligned}(\partial_t + v\cdot\nabla_x + E(t,x)\cdot\nabla_v)&(g(t,x,v) - w(|v|-At))_+\\ &\leq (A - E(t,x)\cdot\frac{v}{|v|})w'(|v|-At)\mathbb{1}_{g(t,x,v)\geq w(|v|-At)},\end{aligned}$$

and the right-hand side is a.e. nonpositive. Therefore, since by assumption

$$g(0,x,v) = |D_x f^{in}(x,v)| + |D_v f^{in}(x,v)| \leq w(|v|) \text{ a.e. } (x,v)\in\mathbb{R}^d\times\mathbb{R}^d,$$

we can similarly conclude that $g(t,x,v) \leq w(|v|-At)$ a.e.
Hence

$$|D_x f(t,x,v)| + |D_v f(t,x,v)| \leq e^{J(t)}w(|v|-At) \text{ a.e.}$$

and in particular

$$|D_x\rho(t,x)| \leq \int_{\mathbb{R}^d} |D_x f(t,x,v)|\mathrm{d}v \leq e^{J(t)}\int_{\mathbb{R}^d} w(|v|-At)\mathrm{d}v \leq R_T e^{J(t)} \text{ a.e.}$$

Step 5. L^∞ estimate for D_xE in terms of $D_x\rho$:
Now we estimate

$$\|D_xE(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = \|\nabla^2 G_d \star \rho(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}$$

where

$$\nabla^2 G_d = \frac{1}{|\mathbb{S}^{d-1}|}\,\mathrm{p.v.}\,\frac{|x|^2 I_d - dxx^\top}{|x|^{d+2}} - \frac{1}{d}\delta_0 I_d.$$

Since the Calderón-Zygmund theory does not apply for $p=\infty$, we instead prove the following lemma:

Lemma (Lemma 4.5.2)**.** *Let Ω be a continuous function on $\mathbb{S}^{d-1}$ which integrates to 0, and let*

$$K := \mathrm{p.v.}\,\frac{\Omega(\frac{x}{|x|})}{|x|^d}.$$

Let $\phi\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ weakly differentiable with $D\phi\in L^\infty(\mathbb{R}^d)$ ($\phi\in W^{1,\infty}(\mathbb{R}^d)$). Then

$$\begin{aligned}\|K\star\phi\|_{L^\infty} &:= \operatorname{ess}\sup_{x\in\mathbb{R}^d}\left|\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^d\setminus B_\varepsilon(0)} \frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)\mathrm{d}y\right|\\ &\leq \|\Omega\|_{L^\infty}(\|\phi\|_{L^1} + |\mathbb{S}^{d-1}|(1+\|\phi\|_{L^\infty}\ln(1+\|D\phi\|_{L^\infty}))).\end{aligned}$$

Proof. We split the domain of integration into three radial regions. Let $0<r<1$, and divide $\mathbb{R}^d\setminus B_\varepsilon(0)$ as

$$\mathbb{R}^d\setminus B_\varepsilon(0) = (\mathbb{R}^d\setminus B_1(0))\sqcup(B_1(0)\setminus B_r(0))\sqcup(B_r(0)\setminus B_\varepsilon(0)).$$

Then fixing $x \in \mathbb{R}^d$, we integrate along each region separately:

$$\int_{\mathbb{R}^d \setminus B_\varepsilon(0)} \frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)\mathrm{d}y = \int_{\mathbb{R}^d\setminus B_1(0)} + \int_{B_1(0)\setminus B_r(0)} + \int_{B_r(0)\setminus B_\varepsilon(0)} \frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)\mathrm{d}y$$
$$=: I_1 + I_2 + I_3.$$

For I_1 we obtain a straightforward L^1 bound:

$$\begin{aligned}|I_1| &\le \int_{\mathbb{R}^d\setminus B_1(0)} \frac{|\Omega(\frac{y}{|y|})|}{|y|^d}|\phi(x-y)|\mathrm{d}y \\ &\le \int_{\mathbb{R}^d\setminus B_1(0)} \|\Omega\|_{L^\infty}|\phi(x-y)|\mathrm{d}y \le \|\Omega\|_{L^\infty}\|\phi\|_{L^1}.\end{aligned}$$

For I_2 we bound both Ω and ϕ by their L^∞ norms and change to spherical coordinates:

$$\begin{aligned}|I_2| &\le \int_{B_1(0)\setminus B_r(0)} \frac{|\Omega(\frac{y}{|y|})|}{|y|^d}|\phi(x-y)|\mathrm{d}y \\ &\le \int_{B_1(0)\setminus B_r(0)} \|\Omega\|_{L^\infty}\|\phi\|_{L^\infty}\frac{\mathrm{d}y}{|y|^d} \\ &\le \|\Omega\|_{L^\infty}\|\phi\|_{L^\infty}\int_r^1 |\mathbb{S}^{d-1}|\frac{R^{d-1}\mathrm{d}R}{R^d} \\ &= \|\Omega\|_{L^\infty}\|\phi\|_{L^\infty}|\mathbb{S}^{d-1}|\ln\frac{1}{r}.\end{aligned}$$

For I_3 we note that, since Ω integrates to 0,

$$\int_{B_r(0)\setminus B_\varepsilon(0)} \frac{\Omega(\frac{y}{|y|})}{|y|^d}\mathrm{d}y = \int_\varepsilon^r \left(\int_{\mathbb{S}^{d-1}} \Omega(u)\mathrm{d}u\right)\frac{R^{d-1}\mathrm{d}R}{R^d} = 0,$$

so we can adjust ϕ in the integral by a constant, namely $\phi(x)$:

$$\begin{aligned}|I_3| &= \left|\int_{B_r(0)\setminus B_\varepsilon(0)} \frac{\Omega(\frac{y}{|y|})}{|y|^d}(\phi(x-y)-\phi(x))\mathrm{d}y\right| \\ &\le \|\Omega\|_{L^\infty}\int_{B_r(0)\setminus B_\varepsilon(0)} \frac{|\phi(x-y)-\phi(x)|}{|y|^d}\mathrm{d}y.\end{aligned}$$

Now because ϕ is weakly differentiable, we have $|\phi(x-y)-\phi(x)| \le \|D\phi\|_{L^\infty}|y|$ a.e. (see Functional Analysis II, lecture 8). Hence, again applying spherical coordinates,

$$\begin{aligned}|I_3| &\le \|\Omega\|_{L^\infty}\int_{B_r(0)\setminus B_\varepsilon(0)} \frac{\|D\phi\|_{L^\infty}|y|}{|y|^d}\mathrm{d}y \\ &= \|\Omega\|_{L^\infty}\|D\phi\|_{L^\infty}|\mathbb{S}^{d-1}|\int_\varepsilon^r \frac{R^{d-1}\mathrm{d}R}{R^{d-1}} \\ &= \|\Omega\|_{L^\infty}\|D\phi\|_{L^\infty}(r-\varepsilon).\end{aligned}$$

Putting the three bounds together, we have

$$\|K\star\phi\|_{L^\infty} \le \|\Omega\|_{L^\infty}(\|\phi\|_{L^1} + |\mathbb{S}^{d-1}|(\|\phi\|_{L^\infty}\ln\frac{1}{r} + \|D\phi\|_{L^\infty}r)).$$

Finally choose

$$r = \frac{1}{1 + \|D\phi\|_{L^\infty}}$$

to obtain

$$\|K \star \phi\|_{L^\infty} \leq \|\Omega\|_{L^\infty}(\|\phi\|_{L^1} + |\mathbb{S}^{d-1}|(\|\phi\|_{L^\infty} \ln(1 + \|D\phi\|_{L^\infty}) + 1)). \qquad \square$$

Then noting that $\nabla^2 G_d$ satisfies the assumptions of the lemma, we have

$$\begin{aligned}\|D_x E(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} &\leq C(1 + \|\phi\|_{L^1} + \|\phi\|_{L^\infty} \ln(1 + \|D\phi\|_{L^\infty}) + 1) \\ &\leq C(1 + \mathcal{M}^{in} + R_T \ln(1 + R_T e^{J(t)})) \\ &\leq C(1 + \mathcal{M}^{in} + R_T \ln((1 + R_T)e^{J(t)})) \\ &\leq C(1 + \mathcal{M}^{in} + R_T \ln(1 + R_T) + R_T J(t))) \leq C_T(1 + J(t))\end{aligned}$$

for some constant $C_T > 0$.

Step 6. Conclusion:

By the estimate on $\|D_x E\|_{L^\infty(\mathbb{R}^d)}$ we have

$$J(t) = \int_0^t (1 + \|D_x E(s, \cdot)\|_{L^\infty(\mathbb{R}^d)}) \mathrm{d}s \leq T(1 + C_T) + C_T \int_0^t J(s) \mathrm{d}s.$$

Therefore by Gronwall's inequality,

$$J(t) \leq T(1 + C_T)e^{TC_T} \text{ for all } t \in [0, T].$$

We then plug this bound back into the previous estimates:

$$\|D_x E(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq C_T(1 + J(t)) \leq C_T(1 + T(1 + C_T)e^{TC_T});$$

$$\|D_x \rho(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq R_T e^{J(t)} \leq R_T \exp\left(C_T(1 + T(1 + C_T)e^{TC_T})\right);$$

and

$$|D_x f(t, x, v)| + |D_v f(t, x, v)| = L(t) \leq e^{J(t)} w(|v| - At) \leq C w(|v| - AT).$$

Therefore we obtain the desired asymptotic behavior for $D_{x,v} f$.

The bound on $D_x E$ then ensures that E is Lipschitz, and this implies that the vector field of the Vlasov equation to which f is a weak solution admits a global flow Z, and the unique weak solution to the Vlasov equation is the pushforward along the flow, i.e. $f = Z\#f^{in}$. Therefore $f \in C^1$, and likewise for ρ.

Finally, $E \in C^1$ follows from elliptic regularity and the fact that $\rho \in C^1$. $\square$