Chapter 2 - Transport Equations

2.1 Transport Eq. with Const. Coeff.

- Recall: Vlasov eq. $\rightarrow$ $\partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) + a(t,x,v)\cdot\nabla_v f(t,x,v) = 0$

Let $v \in \mathbb{R}^n \setminus \{0\}$ and consider following transport eq.

$$\partial_t f + v\cdot\nabla_x f = 0 \qquad (1)$$

where $f \equiv f(t,x) \in \mathbb{R}$ is def. for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$. This is a special case of the Vlasov eq. where $a(t,x,v)=0$ which corresponds ~~to~~ to the "velocity" v beeing const.

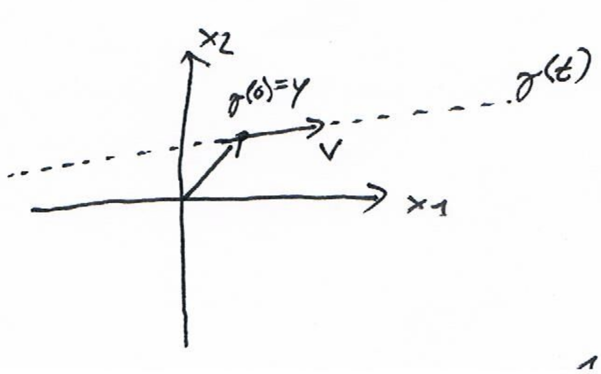
First we define a usefull tool:

- Def. 2.1.1: The characteristic curve of the transport op. $\partial_t + v\cdot\nabla_x$ passing through $y \in \mathbb{R}^n$ at time $t=0$ is the set

$$\{(t,\gamma(t)) \mid t \in \mathbb{R}\}$$

where γ is the solution of the diff. syst. (syst. of characteristics associated to $\partial_t + v\cdot\nabla_x$)

$$\begin{cases} \dot\gamma(t) = v \\ \gamma(0) = y \end{cases}$$



In the case of $v \equiv \text{const.}$ we obviously get

$$\gamma(t) = y + tv \quad , \; t \in \mathbb{R}$$

Hence,

$$\{(t,\gamma(t)) \mid t \in \mathbb{R}\} = \{(t, y+tv) \mid t \in \mathbb{R}\}.$$

Now, we solve (1) with a initial condition (IC)

<u>Thm. 2.1.2:</u> For all $f^{in} \in C^1(\mathbb{R}^n)$ the transport eq with (IC)

$$\begin{cases} \partial_t f(t,x) + v \cdot \nabla_x f(t,x) = 0 & , x \in \mathbb{R}^n, t > 0 \\ f(0,x) = f^{in}(x) \end{cases}$$

has the unique solution $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$ given by

$$f(t,x) = f^{in}(x - tv) \quad , \quad x \in \mathbb{R}^n, \; t \geq 0.$$

<u>Pf:</u> Let's handle the easy part first. Since f^{in} and $\eta : (t,x) \mapsto x - tv$ are both in C^1 we have that their composition $f = f^{in} \circ \eta$ is in C^1. On the other hand we have by the chain rule that

$$\partial_t f(t,x) = \nabla f^{in}(x - tv) \cdot (-v)$$

and

$$\nabla_x f(t,x) = \nabla f^{in}(x - tv).$$

So we get

$$\partial_t f(t,x) + v \cdot \nabla_x f(t,x) = 0, \quad x \in \mathbb{R}^n,\ t>0$$

Clearly we have by definition of f that

$$f(0,x) = f^{in}(x-0) = f^{in}(x).$$

Hence, we have prooven that such fct. exists.

For the uniqueness part we'll see the characteristic curve in action. Suppose $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$ is a solution of (1) and γ is just like in Def. 2.1.1. We know that γ will be of class C^1, so that $t \mapsto f(t,\gamma(t))$ will be of class C^1. We calculate

$$\frac{d}{dt} f(t,\gamma(t)) = \partial_t f(t,\gamma(t)) + \sum_{k=1}^{n} \partial_{x_k} f(t,\gamma(t))\,\dot{\gamma}_k(t)$$

$$= (\partial_t f + v \cdot \nabla_x f)(t,\gamma(t)) = 0, \qquad (2)$$

since $\dot{\gamma}_k(t) = v_k$, $1 \le k \le n$. Therefore

$$f(t,\gamma(t)) = \text{const.}$$

Notice that each solution of (1) is const. along all characteristic curves of $\partial_t + v \cdot \nabla_x$!!!

Now since $t \mapsto f(t, y+tv)$ is const. on $\mathbb{R}_+$ and has to satisfy the (IC) we have

$$f(t, y+tv) = f(0,y) = f^{in}(y), \quad \forall y \in \mathbb{R}^n\ \forall t \ge 0$$

With $x = y+tv$ we get

$$f(t,x) = f^{in}(x-tv) \quad , x \in \mathbb{R}^n, \; t \geq 0.$$

This proves uniqueness and also our theorem.

QED

At last let's see the characteristic curve in another setting:

Example 2.1: Let $a \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$, $S \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$ and $f^{in} \in C^1(\mathbb{R}^n)$ and consider

$$\begin{cases} \partial_t f(t,x) + v \cdot \nabla_x f(t,x) + a(t,x) f(t,x) = S(t,x) \;, x \in \mathbb{R}^n, t > 0 \\ f(0,x) = f^{in}(x) \end{cases}$$

We wanna find a solution. let's consider γ as in Def. 2.1.1. and suppose $f \in C^1(\mathbb{R}_+ + \mathbb{R}^n)$ satisfies the above PDE. As before $g : t \mapsto f(t,\gamma(t))$ is of class C^1 and similarly to (2) we see easily that

$$\frac{d}{dt} f(t,\gamma(t)) = S(t,\gamma(t)) - a(t,\gamma(t)) f(t,\gamma(t))$$

So we have a easy ODE to solve with (IC) $g(t) = f^{in}(y)$.
We know the ~~so~~ unique solution is

$$g(t) = f(t,\gamma(t)) = f^{in}(y) \exp\left(-\int_0^t a(s,\gamma(s))ds\right)$$
$$+ \exp\left(-\int_0^t a(s,\gamma(s))ds\right) \int_0^t S(s,\gamma(s)) \exp\left(\int_0^s a(\tau,\gamma(\tau))d\tau\right) ds$$

With $\gamma(t) = y + tv$ and $x = y + tv$ we can get the solution for $f(t,x)$ as before.