2.2. Transport Equations with Variable Coefficients

We consider the Cauchy problem

$$\begin{cases} \partial_t f(t,x) + V(t,x)\cdot\nabla_x f(t,x) = 0 & \forall t\in(0,T),\ x\in\mathbb{R}^N \\ f(0,x) = f^{in}(x) & \forall x\in\mathbb{R}^N \end{cases}$$

where

$$V:[0,T]\times\mathbb{R}^N\to\mathbb{R}^N \quad \text{and} \quad f^{in}:\mathbb{R}^N\to\mathbb{R}$$

are given, and we assume further that

(H1) V is continuous, differentiable in the second variable, and $\nabla_x V$ is continuous

(H2) $\exists\, k>0$ s.t. $\forall t\in[0,T]$, $x\in\mathbb{R}^N$ $|V(t,x)|\le k(1+|x|)$.

Definition Let $t \in [0,T]$, $x \in \mathbb{R}^N$. If $\gamma : [0,T] \to \mathbb{R}^N$ is a solution to the ODE

$$\begin{cases} \dot\gamma(s) = V(s,\gamma(s)) & \forall s \in (0,T) \\ \gamma(t) = x, \end{cases}$$

then the set $\{(s,\gamma(s)) \mid s \in [0,T]\} \subseteq [0,T] \times \mathbb{R}^N$ is said to be the characteristic curve passing through point x at time t.

Theorem 2.2.2. Suppose V satisfies the assumptions (H1), (H2). Then given $t \in [0,T]$, $x \in \mathbb{R}^N$, the ODE

$$\begin{cases} \dot{\gamma}_{t,x}(s) = V(s, \gamma_{t,x}(s)) & \forall s \in [0,T] \\ \gamma_{t,x}(t) = x \end{cases}$$

has a unique solution $\gamma_{t,x} : [0,T] \to \mathbb{R}^N$.

Set $X(s,t,x) := \gamma_{t,x}(s)$. Then

(a) $X \in C^1([0,T] \times [0,T] \times \mathbb{R}^N ; \mathbb{R}^N)$;

(b) The cross partial derivatives $\partial_s \partial_{x_j} X$, $\partial_{x_j} \partial_s X$ exist, are equal, and are continuous;

(c) If V and $\nabla_x V$ are $k \geq 1$ times cont. differentiable, (H3) then X is $k+1$ times cont. differentiable

Proof Lecture notes, pp. 18.

<u>Counterexample</u> For $N=1$, $V(t,x) = x^2$, condition (H2) is not satisfied.

If γ is a solution to

$$\begin{cases} \dot\gamma(s) = \gamma(s)^2 \\ \gamma(t) = x \end{cases}$$

defined on some interval I containing t, then by separation of variables it must satisfy

$$s-t = \int_t^s d\tau = \int_t^s \underbrace{\frac{1}{\gamma(\tau)^2}\dot\gamma(\tau)}_{1} d\tau = \int_{\gamma(t)}^{\gamma(s)} \frac{1}{y^2}\,dy = \frac{1}{\underbrace{\gamma(t)}_{x}} - \frac{1}{\gamma(s)}$$

$$\Rightarrow \gamma(s) = \frac{1}{1-(s-t)x},$$

which can only be extended to the maximal interval

$$\{ s\in\mathbb{R} \mid (s-t)x < 1 \} \ni t$$

hence not necessarily to $[0,T]$.

<u>Notation</u> $\Phi_t^s : \mathbb{R}^N \to \mathbb{R}^N$; $\Phi_t^s(x) := X(s,t,x)$.

<u>Theorem 2.2.3.</u> Suppose V satisfies (H1), (H2). Then

(a) $\forall t_1, t_2, t_3 \in [0,T]$ $\Phi_{t_2}^{t_3} \circ \Phi_{t_1}^{t_2} = \Phi_{t_1}^{t_3}$; [flow property]

(b) $\forall s,t \in [0,T]$ Φ_t^s is a C^1-diffeomorphism with $(\Phi_t^s)^{-1} = \Phi_s^t$;

(c) $\forall s,t \in [0,T]$, $x \in \mathbb{R}^N$, the function $J_{t,x}(s) := \det D_x \Phi_t^s(x)$ satisfies

$$\begin{cases} \dot{J}_{t,x}(s) = \mathrm{div}_x V(s, \Phi_t^s(x)) \, J_{t,x}(s) \\ J_{t,x}(t) = 1 \end{cases}$$

(d) $\forall s,t \in [0,T]$ Φ_t^s is orientation-preserving; if in addition $\mathrm{div}_x V \equiv 0$, Φ_t^s also preserves the Lebesgue measure i.e. $\forall \phi \in C_c(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \phi \circ \Phi_t^s \, dx = \int_{\mathbb{R}^N} \phi \, dx.$$

Proof

(a) Fix $t_1, t_2 \in [0,T]$, $x \in \mathbb{R}^N$. Then the curve $\gamma_{t_1,x}(s) = \Phi_{t_1}^{s}(x)$ satisfies

$$\begin{cases} \gamma_{t_1,x}(s) = V(s, \gamma_{t_1,x}(s)) \\ \gamma_{t_1,x}(t_2) = \Phi_{t_1}^{t_2}(x) \end{cases}$$

so by uniqueness, $\gamma_{t_1,x} = \gamma_{t_2, \Phi_{t_1}^{t_2}(x)}$ hence

$$\Phi_{t_1}^{s}(x) = \Phi_{t_2}^{s}(\Phi_{t_1}^{t_2}(x)).$$

(b) $\Phi_t^t = \mathrm{id}_{\mathbb{R}^N}$ since $\Phi_t^t(x) = \gamma_{t,x}(t) = x$, so by (a)

$$\Phi_t^s \circ \Phi_s^t = \Phi_s^s = \mathrm{id}_{\mathbb{R}^N} = \Phi_t^t = \Phi_s^t \circ \Phi_t^s$$

so $(\Phi_t^s)^{-1} = \Phi_s^t$ and Φ_t^s is a C^1-diffeo.

(c) The smooth function $\det : M_N(\mathbb{R}) \to \mathbb{R}$ has differential $D\det : M_N(\mathbb{R}) \to \mathbb{R}$ given by the linear map $(D\det)_A B = \det(A) \cdot tr(B A^{-1})$ for $A \in GL_N(\mathbb{R})$, $B \in M_N(\mathbb{R})$.

So by the chain rule,

$$j_{t,x}(s) = (D\det)_{\underbrace{D_x \Phi_t^s(x)}_{\in GL_N(\mathbb{R})}} \partial_s D_x \Phi_t^s(x) = \underbrace{\det(D_x \Phi_t^s(x))}_{J_{t,x}(s)} \cdot tr(\partial_s D_x \Phi_t^s(x)(D_x \Phi_t^s(x))^{-1}).$$

We now calculate

$$\partial_s D_x \Phi_t^s(x) \overset{\text{Thm 2.2.2.(b)}}{=} D_x \partial_s \underbrace{\Phi_t^s(x)}_{\gamma_{t,x}(s)} = D_x\left[V(s, \Phi_t^s(x))\right]$$

$$= (D_x V)(s, \Phi_t^s(x)) D_x \Phi_t^s(x) \quad \text{by the chain rule}$$

$$\Rightarrow tr(\partial_s D_x \Phi_t^s(x)(D_x \Phi_t^s(x))^{-1}) = tr(D_x V(s, \Phi_t^s(x))) = div_x V(s, \Phi_t^s(x)).$$

And $J_{t,x}(t) = \det D_x \Phi_t^t(x) = \det id_{\mathbb{R}^N} = 1.$

(d) $J_{t,x}$ is continuous and always nonzero, so it must always have the same sign. Since $J_{t,x}(t) = 1$, this implies $J_{t,x}(s) > 0$ $\forall s, t \in [0,T]$, $x \in \mathbb{R}^N$.

Hence Φ_t^s is always orientation-preserving.

Furthermore, if $\operatorname{div}_x V \equiv 0$, $J_{t,x}$ is constant hence identically 1.

Thus by change of variables $\forall \phi \in C_c(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \phi \, dx = \int_{\Phi_t^s(\mathbb{R}^N)} \phi \, dx = \int_{\mathbb{R}^N} \phi \circ \Phi_t^s \cdot \underbrace{|\det(D_x \Phi_t^s)|}_{1} \, dx$$

$$= \int_{\mathbb{R}^N} \phi \circ \Phi_t^s \, dx$$

hence in this case, Φ_t^s is also measure-preserving.

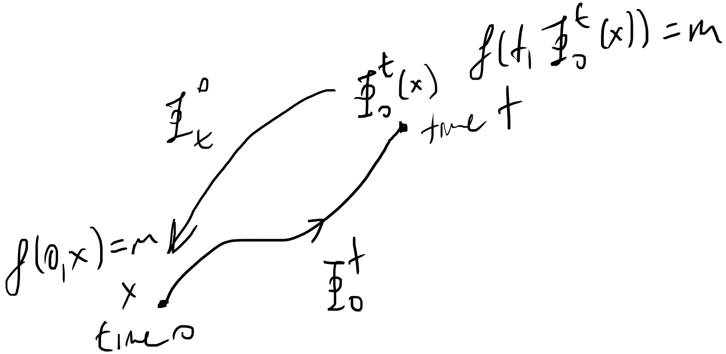
Theorem 2.2.4 Suppose V satisfies (H1), (H2), and $f^{in} \in C^1(\mathbb{R}^N)$.

Then the transport equation

$$\begin{cases} \partial_t f(t,x) + V(t,x)\cdot\nabla_x f(t,x) = 0 \\ f(0,x) = f^{in}(x) \end{cases}$$

has the unique solution $f : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ given by

$$f(t,x) = f^{in}(\Phi_t^0(x)), \qquad t \in [0,T],\ x \in \mathbb{R}^N.$$



Proof

- Uniqueness: Suppose f is such a solution. Define

$$\rho(t,x) := f(t, \underbrace{\Phi_0^t(x)}_{\text{characteristic curve}}).$$

Then

$$\partial_t \rho(t,x) = (\partial_t f)(t, \Phi_0^t(x)) + \nabla_x f(t, \Phi_0^t(x)) \cdot \overbrace{\underbrace{\partial_t \Phi_0^t(x)}_{\gamma_{0,x}(t)}}^{V(t, \Phi_0^t(x))}$$

$$= (\partial_t f + V \cdot \nabla_x f)(t, \Phi_0^t(x)) = 0.$$

Therefore ρ is constant with respect to t, hence $\forall t \in [0,T], x \in \mathbb{R}^n$

$$f(t,x) = f(t, \Phi_0^t(\Phi_t^0(x))) = \rho(t, \Phi_t^0(x)) = \rho(0, \Phi_t^0(x)) = f(0, \Phi_t^0(x)) = f^{in}(\Phi_t^0(x)).$$

Therefore if f is a solution, it must coincide with $f^{in}(\Phi_t^0(x))$.

- Existence: define $f(t,x) = f^{in}(\Phi_t^0(x))$, then $f(0,x) = f^{in}(x)$ and

$$\partial_t f(t,x) = \nabla_x f^{in}(\Phi_t^0(x)) \cdot \partial_t \Phi_t^0(x)$$

$$\nabla_x f(t,x) = \underbrace{\nabla_x f^{in}(\Phi_t^0(x))}_{\substack{\text{row vector} \\ \mathbb{R}^{1\times n}}} \cdot \underbrace{D_x \Phi_t^0(x)}_{\substack{n\times n \text{ matrix} \\ \mathbb{R}^{n\times n}}}$$

so $(\partial_t f + V\cdot\nabla_x f)(t,x) = \nabla_x f^{in}(\Phi_t^0(x)) \cdot (\partial_t \Phi_t^0(x) + D_x \Phi_t^0(x)\, V(t,x))$.

Now differentiate $\Phi_t^0 \circ \Phi_0^t(x) = x$ with respect to t:

$$0 = \partial_t \Phi_t^0(\Phi_0^t(x)) + D_x \Phi_t^0(\Phi_0^t(x))\, \underbrace{\partial_t \Phi_0^t(x)}_{V(t,\Phi_0^t(x))}$$

$$= (\partial_t \Phi_t^0 + D_x \Phi_t^0 \cdot V)(t, \Phi_0^t(x)).$$

Then for each $y \in \mathbb{R}^n$, substituting $x = \Phi_t^0(y)$ yields

$$\partial_t \Phi_t^0(t,y) + D_x \Phi_t^0(t,y)\, V(t,y) = 0$$

hence

$$\partial_t f(t,y) + V(t,y)\cdot\nabla_x f(t,y) = \nabla_x f^{in}(\Phi_t^0(y)) \cdot 0 = 0.$$

$\square$.