# CONSERVATIVE TRANSPORT AND WEAK SOLUTIONS (PART I)

CAMILLA NAIARETTI

## Introduction

This text has two main goals: the first is to review some notion of measure theory, in particular the concept of transportation of measures, and the second is to define the weak solution of the Cauchy problem for the conservative transport equation and relate it with the classical notion of solution.

The notion of weak solution of a PDE can be most easily defined within the theory of distribution. In the case of a 1st order PDE, especially in the context of statistical mechanics, it is convenient to consider weak solutions that are measures, instead of more general distributions. For this reason, we have to review the notion of transportation of measures.

## 1. Transportation of measures

**Definition 1.** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces ($\mathcal{A}$ and $\mathcal{B}$ are respectively the $\sigma$-algebras of $X$ and $Y$). Let $T : X \to Y$ be an $(\mathcal{A}, \mathcal{B})$-measurable map, and let $\mu$ be a positive measure on $(X, \mathcal{A})$. Then a positive measure on $(Y, \mathcal{B})$ is defined as

$$\nu(B) \coloneqq \mu(T^{-1}(B)),$$

where $B \subseteq Y$.
This measure is known as "the push-forward of the measure $\mu$ under the map $T$" and is denoted by

$$\nu \coloneqq T\#\mu.$$

**Remark 1**. An equivalent definition of $\nu \coloneqq T\#\mu$ is

$$\int_Y \mathbb{1}_B(y)\nu(dy) = \int_X \mathbb{1}_{T^{-1}(B)}(x)\mu(dx) = \int_X \mathbb{1}_B(T(x))\mu(dx), \tag{1}$$

where we use that

$$\mathbb{1}_{T^{-1}(B)} = \mathbb{1}_B \circ T.$$

The formula (1) in the remark can be generalized in the following proposition.

*Date*: March 2021.

**Proposition 1**. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, let $T : X \to Y$ be an $(\mathcal{A}, \mathcal{B})$-measurable map, and let $\mu$ be a positive measure on $(X, \mathcal{A})$. Set $\nu := T\#\mu$. Then

$$\phi \in L^1(Y, \nu) \Rightarrow \phi \circ T \in L^1(X, \mu)$$

and

$$\int_Y \phi(y)\nu(dy) = \int_X \phi(T(x))\mu(dx).$$

*Proof.* We divide the proof in four steps. This kind of proof is sometimes called "measure theoretical induction".

(a) Let $\phi = \mathbb{1}_B$ with $B \in \mathcal{B}$. By the remark above we have

$$\int_Y \phi(y)\nu(dy) = \int_Y \mathbb{1}_B(y)\nu(dy) = \int_X \mathbb{1}_B(T(x))\mu(dx) = \int_X \phi(T(x))\mu(dx).$$

So the statement holds for all $\phi$ of the form $\phi = \mathbb{1}_B$.

(b) Let $\phi = \sum_{i=1}^N b_i \mathbb{1}_{B_i}$ with $B_i \in \mathcal{B}$ and $b_i \in \mathbb{R}$. By linearity and (a) we conclude that the equation holds for all simple functions.

(c) Let $\phi \in L^1(Y, \nu)$ such that $\phi \geq 0$. Then there exists a sequence $(\phi_n)_{n \geq 0}$ of simple function such that $\phi = \sup_n \phi_n$. By monotone convergence theorem and (b) we get

$$\begin{aligned}
\int_Y \phi(y)\nu(dy) &= \int_Y \sup_n \phi_n(y)\nu(dy) \\
&= \sup_n \int_Y \phi_n(y)\nu(dy) \\
&= \sup_n \int_X \phi_n(T(x))\mu(dx) \\
&= \int_X \sup_n \phi_n(T(x))\mu(dx) \\
&= \int_X \phi(T(x))\mu(dx).
\end{aligned}$$

(d) For arbitrary $\phi \in L^1(Y, \nu)$, we write

$$\phi = \phi^+ - \phi^-,$$

where $\phi^+ = \max(\phi, 0)$ and $\phi^- = \max(-\phi, 0)$. $\phi^+, \phi^- \geq 0$ and $\phi^1, \phi^- \in L^1(Y, \nu)$. By (c) the formula follows for $\phi^+$ and $\phi^-$. Then, by linearity of the integral we conclude that the formula holds also for $\phi$.

This concludes the proof.

□

We recap some concepts of measure theory. Intuitively, we will work with measures, as densities can be ill-defined. For instance, if the whole mass is concentrated at a single point, then a "nice" density representation fails to exist. Hence, here and henceforth, we work with a measure instead of the density.

## 2. Weak solution of the Cauchy problem for the conservative transport equation

In the following, we define weak solutions of the transport equation. We will see that when the density of the system is "sufficiently nice", that is the derivative of the density is well-defined ($f \in C^1([0,T] \times \mathbb{R}^N)$) and the mass is not concentrated in a point, then the weak solution and the classical notion of solution are related (see proposition 2). Otherwise, in general, we use the weak solution when we are dealing with a "complicated transport equation" (i.e., mass concentrated in a point and consequently the density is not $C^1$).

**Definition 2.** Let $V \in C([0,T] \times \mathbb{R}^N)$, and let $\mu^{in} \in \mathcal{M}(\mathbb{R}^N)$. A weak solution of the Cauchy problem for the conservative transport equation

$$\begin{cases} \partial_t \mu + \mathrm{div}_x(\mu V) = 0 \\ \mu\Big|_{t=0} = \mu^{in}, \end{cases}$$

is an element $\mu$ of $C([0,T]; w - \mathcal{M}(\mathbb{R}^N))$ that satisfies the initial condition and the equality

$$\int_0^T \int_{\mathbb{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x)) \mu(t,dx) dt = 0,$$

for each so-called test function $\phi \in C_c^1((0,T) \times \mathbb{R}^N)$.

The following proposition shows the relation between the notion of weak solution and the classical notion of solution of the conservative transport equation. In particular, it establishes consistency: when the transport equation is "sufficiently nice" (i.e., mass not concentrated in a point and derivative of the density well-defined), then the two notions of solution coincide.

**Proposition 2**. Let $f \in C^1([0,T] \times \mathbb{R}^N)$. Then

$$\partial_t f + \mathrm{div}_x(fV) = 0 \quad on \quad [0,T] \times \mathbb{R}^N$$

if and only if

$$\int_0^T \int_{\mathbb{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x)) f(t,x) dx dt = 0,$$

for each $\phi \in C_c^1((0,T) \times \mathbb{R}^N)$.

*Proof.* Let us consider the vector field

$$W : (t,x) \mapsto (\phi f, \phi f V)(t,x)$$

defined on $[0,T] \times \mathbb{R}^N$ with values in $\mathbb{R} \times \mathbb{R}^N$. Since $W$ is a product of well-defined functions, $W$ is well-defined. In addition, since $\phi$ has compact support, by construction $W \in C_c^1((0,T) \times \mathbb{R}^N; \mathbb{R} \times \mathbb{R}^N)$. Therefore, we can apply Gauss theorem in the domain $(0,T) \times \mathbb{R}^N$: the integral of the divergence of the vector field equals to the integral on the boundary of the vector field; but since $W$ has compact support, its integral on the boundary is 0. Hence, we get

$$0 = \int \int_{(0,T) \times \mathbb{R}^N} \mathrm{div}_{t,x}(\phi f, \phi f V)(t,x) dx dt.$$

By the product rule of derivatives, the above equation can be rewritten as

$$
\begin{aligned}
0 = &\int\int_{(0,T)\times\mathbb{R}^N} [\partial_t\phi(t,x) + V(t,x)\cdot\nabla_x\phi(t,x)]f(t,x)dxdt \\
&+ \int\int_{(0,T)\times\mathbb{R}^N} \phi(t,x)[\partial_t f(t,x) + \mathrm{div}_x(f(t,x)V(t,x))]dxdt.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int\int_{(0,T)\times\mathbb{R}^N} [\partial_t\phi(t,x) + V(t,x)\cdot\nabla_x\phi(t,x)]f(t,x)dxdt \\
&= -\int\int_{(0,T)\times\mathbb{R}^N} \phi(t,x)[\partial_t f(t,x) + \mathrm{div}_x(f(t,x)V(t,x))]dxdt,
\end{aligned}
\tag{2}
$$

for each $\phi \in C_c^1((0,T)\times\mathbb{R}^N)$. With this preparation we can now prove the two directions of the proposition.

"$\Rightarrow$": If $\partial_t f + \mathrm{div}_x(fV) = 0$ on $[0,T]\times\mathbb{R}^N$, then the integral on the right hand side of (2) is 0 for all $\phi \in C_c^1((0,T)\times\mathbb{R}^N)$. Hence,

$$
\int\int_{(0,T)\times\mathbb{R}^N} [\partial_t\phi(t,x) + V(t,x)\cdot\nabla_x\phi(t,x)]f(t,x)dxdt = 0,
$$

for each $\phi \in C_c^1((0,T)\times\mathbb{R}^N)$. This means that $f$ is a weak solution of the transport equation in $(0,T)\times\mathbb{R}^N$.

"$\Leftarrow$": Conversely, if the integral on the left hand side of (2) is 0 (i.e., $f$ is a weak solution of the transport equation in $(0,T)\times\mathbb{R}^N$) for all $\phi \in C_c^1((0,T)\times\mathbb{R}^N)$, by (2), we must have

$$
\int\int_{(0,T)\times\mathbb{R}^N} \phi(t,x)[\partial_t f(t,x) + \mathrm{div}_x(f(t,x)V(t,x))]dxdt = 0,
$$

for each $\phi \in C_c^1((0,T)\times\mathbb{R}^N)$. Since $\phi$ is arbitrary, we conclude that

$$
\partial_t f + \mathrm{div}_x(fV) = 0 \quad \text{on} \quad [0,T]\times\mathbb{R}^N. \qquad \square
$$

## References

[1] F.Golse. *Mean Field Kinetic Equations.* 2013. Chap. II, §3.