Conservative transport equation

Topics in Non-Collisional Kinetic Theory

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Outline

Cauchy problem

By the Riesz-Markov-Kakutani theorem, we associate a Radon measure μ to our positive distribution f.

- A Radon measure μ^{in} representing an initial positive distribution
- A continuous vector field $V \in C([0, T] \times \mathbb{R}^N)$
- Cauchy problem for the conservative transport equation

$$\begin{cases} \partial_t \mu + \operatorname{div}(\mu V) = 0, \\ \mu_{|t=0} = \mu^{\text{in}}. \end{cases}$$

Integral equation

We investigate the so called 'conservative' transport equation.

- Sufficiently well-behaved distribution f
- The transport equation takes an integral from

$$\frac{d}{dt}\int_{\Omega} f(t,x)dx = \int_{\partial\Omega} f(t,x)V(t,x)\cdot dS(x)$$

Hypothesis

We use up to three hypotheses that go in two categories: regularity and estimates.

- (H1) - $V \in C([0,T] \times \mathbb{R}^N; \mathbb{R}^N)$ and $\nabla_x V \in C([0,T] \times \mathbb{R}^N; M_N(\mathbb{R}))$
- (H2) - $|V(t,x)| \leq \kappa(1+|x|)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$
- (H3) - $V \in C^k([0,T] \times \mathbb{R}^N; \mathbb{R}^N)$ and $\nabla_x V \in C^k([0,T] \times \mathbb{R}^n; M_N(\mathbb{R}))$

Existence and uniqueness result for weak solution

Theorem

Let V satisfy (H1) and (H2), and let μ^{in} be a positive radon measure. Then the Cauchy problem for the conservative transport equation has a unique weak solution μ given by the formula

$$\mu(t) = X(t, 0, \cdot)\#\mu^{in}, \quad t \in [0, T].$$

In particular $\mu(t)$ is a positive radon measure for all $t \in [0, T]$, and μ is bounded if μ^{in} is bounded. Moreover,

$$\int_{\mathbb{R}^N} \mu(t, dx) = \int_{\mathbb{R}^N} \mu^{in}(dx).$$

Existence

- The flow $X(t, 0, \cdot)$ is a C^1-diffeomorphism
- The map $t \mapsto \int_{\mathbb{R}^N} \varphi(t, X(t, 0, y))\mu^{in}(dy)$ is of class C^1
- Construct the pushforward measure $\mu(t) = X(t, 0, \cdot)\#\mu^{\text{in}}$
- Verify that μ is a weak solution

$$
\begin{aligned}
0 &= \left[\int_{\mathbb{R}^N} \varphi(t, X(t, 0, y))\mu^{\text{in}}(dy)\right]_0^T \\
&= \int_0^T \left(\frac{d}{dt}\int_{\mathbb{R}^N} \varphi(t, X(t, 0, y))\mu^{\text{in}}(dy)\right) dt
\end{aligned}
$$

Existence

- Compute the derivative of $t \mapsto \int_{\mathbb{R}^N} \varphi(t, X(t,0,y))\mu^{in}(dy)$

$$\frac{d}{dt}\int_{\mathbb{R}^N} \varphi(t, X(t,0,y))\mu^{\text{in}}(dy)$$
$$= \int_{\mathbb{R}^N} (\partial_t \varphi(t, X(t,0,y)) + \nabla_x \varphi(t, X(t,0,y)) \cdot V(t, X(t,0,y)))\mu^{\text{in}}(dy)$$

- Use the integral characterisation of the pushforward measure

$$\frac{d}{dt}\int_{\mathbb{R}^N} \varphi(t, X(t,0,y))\mu^{\text{in}}(dy)$$
$$= \int_{\mathbb{R}^N} (\partial_t \varphi(t,x) + \nabla_x \varphi(t,x) \cdot V(t,x))\mu(t,dx)$$

- μ is a weak solution

$$0 = \int_0^T \left(\int_{\mathbb{R}^N} (\partial_t \varphi(t,x) + \nabla_x \varphi(t,x) \cdot V(t,x))\mu(t,dx) \right) dt$$

Uniqueness

- Weak solution of the Cauchy problem μ exists
- Construct the pushforward measure $\nu(t) = X(0, t, \cdot)\#\mu(t)$
- The map $(t, x) \mapsto \phi(X(0, t, x))$ is of class C^1 where $\phi \in C_c^1(\mathbb{R}^N)$ and satisfies

$$(\partial_t + V(t, x) \cdot \nabla)\phi(X(0, t, x)) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^N$$

- Set $\Psi(t, x) = \varphi(t)\phi(X(0, t, x))$ and compute the derivative of $t \mapsto \int_{\mathbb{R}^N} \phi(x)\nu(t, dx)$

Uniqueness

- Suppose Ψ has compact support in $(0, T) \times \mathbb{R}^N$

$$
\begin{aligned}
& -\int_0^T \varphi'(t) \left(\int_{\mathbb{R}^N} \phi(x)\nu(t, dx) \right) dt \\
& = -\int_0^T \left(\int_{\mathbb{R}^N} \varphi'(t)\phi(X(0, t, y))\mu(t, dy) \right) dt \\
& = -\int_0^T \left(\int_{\mathbb{R}^N} \varphi'(t)\phi(X(0, t, y))\mu(t, dy) \right) dt \\
& \quad - \int_0^T \left(\int_{\mathbb{R}^N} \underbrace{(\partial_t + V(t, y) \cdot \nabla)\phi(X(0, t, y))}_{=0} \mu(t, dy) \right) dt \\
& = -\int_0^T \left(\int_{\mathbb{R}^N} (\partial_t + V(t, y) \cdot \nabla_x)\Psi(t, y)\mu(t, dy) \right) dt \\
& = 0
\end{aligned}
$$

Uniqueness

- The map $t \mapsto \int_{\mathbb{R}^N} \phi(x)\nu(t, dx)$ is constant

$$\int_{\mathbb{R}^N} \phi(x)\nu(t, dx) = \int_{\mathbb{R}^N} \phi(x)\nu(0, dx) = \int_{\mathbb{R}^N} \phi(x)\mu^{\text{in}}(dx)$$

- The flow X satisfies $X(t, s, \cdot) \circ X(s, t, \cdot) = Id_{\mathbb{R}^N}$
- $\mu(t) = X(t, 0, \cdot)\#\mu^{\text{in}}$; the weak solution is unique

Support of Ψ

- Estimate for the flow X

$$|X(s,t,x)| \leq (|x| + \kappa T)e^{\kappa T}, \quad (s,t,x) \in [0,T]^2 \times \mathbb{R}^N$$

- The test function ϕ has $\mathrm{supp}(\phi) \subset B(0,R)$
- The composition $\phi \circ X(0,t,\cdot)$ has

$$\mathrm{supp}(\phi \circ X(0,t,\cdot)) \subset B(0,(R+\kappa T)e^{\kappa T})$$

- Ψ has compact support in $(0,T) \times \mathbb{R}^N$

Regularity result with C^1 initial data

Theorem

Let V satisfy (H1), (H2), and (H3) (case $k = 1$), and let f^{in} be of class C^1. Then the Cauchy problem for the conservative transport equation has a unique classical solution $f \in C^1([0, T] \times \mathbb{R}^N)$ given by the formula

$$f(t,x) = f^{in}(X(0,t,x))J(0,t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^N.$$

In particular, $f(t,\cdot)$ is integrable for all $t \in [0,T]$ if f^{in} is integrable, and

$$\int_{\mathbb{R}^N} f(t,x)dx = \int_{\mathbb{R}^N} f^{in}(x)dx.$$

Uniqueness

- Use linearity to reduce to the case where if $g \in C^1([0, T] \times \mathbb{R}^N$ satisfies the following problem, then it is zero map:

$$\begin{cases} \partial_t g(t,x) + \mathrm{div}(g(t,x)V(t,x)) = 0, \\ g_{|t=0} = 0. \end{cases}$$

- Compute the derivative of $t \mapsto g(t, X(t,0,x))$

$$\begin{aligned} \frac{d}{dt} g(t, X(t,0,x)) &= (\partial_t g + V \cdot \nabla_x g)(t, X(t,0,x)) \\ &= -g(t, X(t,0,x)) \mathrm{div}_x V(t, X(t,0,x)) \end{aligned}$$

- Deduce that g is the zero map
- The solution is unique

Recover classical solution from weak solution

- $J(s,t,x) = \det(\nabla_x X(s,t,x))$
- Split $f^{\text{in}} = f_1^{\text{in}} - f_2^{\text{in}}$ with $f_1^{\text{in}} = \sqrt{1 + (f^{\text{in}})^2}$
- Construct the pushforward measures $\mu_{1,2}(t) = X(t,0,\cdot)\#(f_{1,2}^{\text{in}}\mathscr{L}^N)$
- Compute the pushforward measures

$$\begin{aligned}\int_\Omega h(x)\mu_{1,2}(t,dx) &= \int_{X(t,0,\Omega)} h(X(t,0,y))f_{1,2}^{\text{in}}(y)\mathscr{L}^N(dy)\\ &= \int_\Omega h(x)f_{1,2}^{\text{in}}(X(t,0,x)^{-1})|\det(\nabla_x X(t,0,y)|^{-1}\mathscr{L}^N(dx)\\ &= \int_\Omega h(x)f_{1,2}^{\text{in}}(X(t,0,x)^{-1})|\det(\nabla_x X(t,0,x)^{-1})|\mathscr{L}^N(dx)\end{aligned}$$

- Deduce that $\mu_{1,2}(t) = f_{1,2}^{\text{in}}(X(0,t,\cdot))J(0,t,\cdot)\mathscr{L}^N = f_{1,2}(t,\cdot)\mathscr{L}^N$
- f is of class C^1 and thus is a classical solution