From particle systems to mean field PDE's

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March 29, 2021

Goal

We want to solve a for the evolution of N particles which interact according to an interaction kernel $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. In other words, the evolution of the N particles satisfy

$$\forall i \in [N] : y_i'(t) = \sum_{\substack{j=1 \\ j \neq i}}^{N} K(y_i(t), y_j(t))$$

We assume $K \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ and require the existence of a Lipschitz constant $L \geqslant 1$ such that $\forall x, y \in \mathbb{R}^d$

1. $\left|\sup_{y \in \mathbb{R}^d} \nabla_x K(x,y)\right| \leqslant L$, $\left|\sup_{x \in \mathbb{R}^d} \nabla_y K(x,y)\right| \leqslant L$
2. $K(y,x) = -K(x,y)$

The first point implies
$\forall x, y \in \mathbb{R}^d : |K(x,y)| \leqslant |K(0,0)| + L(|x| + |y|)$.

Lemma

$\mathcal{K} : \mathcal{P}_1(\mathbb{R}^d) \to Lip(\mathbb{R}^d, \mathbb{R}^d); \mu \mapsto \mathbb{R}^d \to \mathbb{R}; x \mapsto \int_{\mathbb{R}^d} K(x,y)d\mu(y)$ *is well defined where* $\mathcal{P}_1(\mathbb{R}^d) := \{\mu \in \mathcal{P}(\mathbb{R}) | \int_{\mathbb{R}^d} |x| \, d\mu(x) < \infty\}$

Proof.

We compute

$$\left| \int_{\mathbb{R}^d} K(x,y)d\mu(y) \right| \leqslant \int_{\mathbb{R}^d} |K(x,y)| \, d\mu(y) \leqslant \int_{\mathbb{R}^d} |K(0)| + L(|x| + |y|)d\mu(y)$$

$$\leqslant |K(0)| + L\,|x|\,\mu(\mathbb{R}^d) + L \int_{\mathbb{R}^d} |y| \, d\mu(y) < \infty$$

$$\left| \int_{\mathbb{R}^d} K(x,y) - K(z,y)d\mu(y) \right| \leqslant \int_{\mathbb{R}^d} L\,|x-z| \, d\mu(y) = L\,|x-z|\,\mu(\mathbb{R}^d) < \infty$$

□

Rescaling

Set

$$s : \mathbb{R} \to \mathbb{R}; t \to \frac{t}{N}$$

$$z_i := y_i \circ s$$

Then

$$\begin{aligned} z_i'(t) &= y_i'(s(t))\frac{1}{N} \\ &= \frac{1}{N}\sum_{j=1:j\neq i}^{N} K(y_i(s(t)), y_j(s(t))) \\ &= \frac{1}{N}\sum_{j=1:j\neq i}^{N} K(z_i(t), z_j(t)) \end{aligned}$$

We can recover $y_i(t) = y_i(s(Nt)) = z_i(Nt)$.

Now we can bound $z_i'(t)$ in terms of K independent of N and it becomes sensible to assume the existence of a measure $\mu(t)$ on $\mathbb{R}$ for every t such that

$$\lim_{n\to\infty} \frac{1}{N} \sum_{j=1}^{N} K(z_i(t), z_j(t)) = \int_{\mathbb{R}^d} K(z_i(t), y) d\mu(t)(y)$$

If we thus have infinitely many particles desitributed according to $\mu(t)$ at time t we get a new differential equation.

$$z'(t) = \int_{\mathbb{R}^d} K(z(t), y) d\mu(t)(y)$$

For $\phi \in C_c^1(\mathbb{R}^d, \mathbb{R})$ and N particles we have

$$d_t \frac{1}{N} \sum_{j=1}^{N} \phi(z_i(t)) = \frac{1}{N} \sum_{j=1}^{N} \nabla\phi(z_i(t)) z_i'(t)$$

$$= \frac{1}{N} \sum_{j=1}^{N} \nabla\phi(z_i(t)) \mathcal{K}(f(t, \cdot))(z_i(t))$$

If we instead assume a general distribution we arrive at
$d_t \int\limits_{\mathbb{R}^d} \phi(z) d\mu(t, z) = \int\limits_{\mathbb{R}^d} \nabla\phi(z) \cdot \mathcal{K}(\mu(t, z)) d\mu(t, z)$
This mean field equation can be consisely denoted by

$$\partial_t \mu + div_z(\mu \mathcal{K}(\mu)) = 0$$

.

Lemma

Let $U \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3, \mathbb{R})$ be the potential and $f \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^3)$ a solution of the Poisson-Vlasov model such that

$$\Delta_x U(t,x) = \frac{q}{\epsilon_o} \int\limits_{\mathbb{R}^3} f(t,x,v)dv$$

Define $K(x,v,y,w) = (v - w, \frac{q^2}{4\pi m \epsilon_0} \frac{x-y}{|x-y|})$. Assume

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} f(0,x,v)dxdv = 1 \qquad \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} v \cdot f(0,x,v)dxdv = 0$$

Then

$$\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} K(x,v,y,w)f(t,y,w)dydw = (v, \frac{-q}{m}\nabla_x U(t,x))$$

In particular, this almost fits into our framework, except that the kernel is singular on $\{(x,v,y,w) \in (\mathbb{R}^3)^4 | x = y\}$.

Proof

We calculate

$$d_t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v)dxdv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \partial_t f(t,x,v)dxdv$$

$$= -\int_{\mathbb{R}^3} v \cdot \int_{\mathbb{R}^3} \nabla_x f(t,x,v)dxdv - \int_{\mathbb{R}^3} F(t,x) \cdot \int_{\mathbb{R}^3} \nabla_v f(t,x,v)dvdx = 0$$

Hence $\forall t \in \mathbb{R}: \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,x,v)dxdv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(0,x,v)dxdv = 1$

$$d_t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v \cdot f(t,x,v)dxdv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v \cdot \partial_t f(t,x,v)dxdv$$

$$= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v \cdot (v \cdot \nabla_x f(t,x,v))dxdv - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v \cdot (F(t,x) \cdot \nabla_v f(t,x,v))dxdv$$

$$= 0$$

where we use that

$$\int_{\mathbb{R}^3} v \cdot (v \cdot \nabla_x f(t,x,v))dx = \sum_{i,j=1}^{d} v_i v_j \int_{\mathbb{R}^3} \partial_{x_j} f(t,x,v)dx = 0$$

as well as the next computation

$$-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v \cdot (F(t,x) \cdot \nabla_v f(t,x,v))dxdv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla_v (v \cdot F(t,x)) f(t,x,v)dxdv$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} F(t,x) f(t,x,v)dxdv = -\int_{\mathbb{R}^3} \nabla_x U(t,x) \int_{\mathbb{R}^3} f(t,x,v)dvdx = -\int_{\mathbb{R}^3} \nabla_x U(t,x)\rho(t,x)dx$$

$$= \frac{\epsilon_0}{q} \int_{\mathbb{R}^3} \nabla_x U(t,x) \Delta_x U(t,x)dx = 0$$

To calculate $\mathcal{K}$ we observe

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (v-w) f(t,y,w)dydw = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(t,y,w)dydw \cdot v - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} w f(t,y,w)dydw = v \cdot 1 - 0 = v$$

Fundamental solution of Laplace operator

Let $G(x) = \frac{1}{4\pi|x|}$ be the fundamental solution with derivatives

$$\partial_i G(x) = \frac{x_i}{4\pi\left|x\right|^3}$$

G as well as its first derivatives are locally integrable but not continuous at 0.
We have

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{q^2}{4\pi\epsilon_0 m}\frac{x-y}{\left|x-y\right|^3} f(t,y,w)dydw &= \frac{q}{m}\int_{\mathbb{R}^3} \frac{x-y}{4\pi\left|x-y\right|^3}\int_{\mathbb{R}^3}\frac{q}{\epsilon_0} f(t,y,w)dwdy \\
&= \frac{q}{m}\int_{\mathbb{R}^3} \nabla G(x-y)\Delta_y U(t,y)dy \\
&= \frac{-q}{m}\int_{\mathbb{R}^3} G(x-y)\Delta_y \nabla_y U(t,y)dy \\
&= \frac{-q}{m}\nabla_x U(t,x)
\end{aligned}$$

Theorem 3.2.1

Let $Z = (Z_i)_{i\in[N]} \in (\mathbb{R}^d)^N$ be the starting points and denote

$$\mu_Z = \frac{1}{N}\sum_{i=1}^{N}\delta_{Z_i}$$

the empirical measure. There is a unique solution $(z_i)_{i\in[N]} \in C^1(\mathbb{R}, (\mathbb{R}^d)^N)$ to the initial value problem

$$\begin{cases} z_i'(t) = \frac{1}{N}\sum_{j=1}^{N} K(z_i(t), z_j(t)) \\ (z_i(0))_{i\in[N]} = Z \end{cases}$$

Furthermore at time t the empirical measure $\mu_{(z_i(t))_{i\in[N]}} = \frac{1}{N}\sum_{i=1}^{N}\delta_{z_i(t)}$ is a weak solution of

$$\begin{cases} \partial_t\mu(t) + div_x(\mu\mathcal{K}(\mu(t))) = 0 \\ \mu(0) = \mu_Z \end{cases}$$

Proof

The function

$$f : \mathbb{R} \times (\mathbb{R}^d)^N \to (\mathbb{R}^d)^N; (t, (x_i)_{i\in[N]}) \mapsto (\frac{1}{N}\sum_{j=1}^{N} K(x_i, x_j))_{i\in[N]}$$

is continuous, Lipschitz in the location, and grows at most linearly. Hence the Picard-Lindelöf Theorem gives us the first claim.

Define $\mu(t) := \mu_{(z_j(t))_{j\in[N]}} = \frac{1}{N}\sum_{j=1}^{N} \delta_{z_j(t)}$

Let $\phi \in C_c^1((0, T) \times \mathbb{R}^d, \mathbb{R})$ and calculate

$$\mathcal{K}(\mu(t))(x) = \int_{\mathbb{R}^d} K(x, z) d\mu(t)(z) = \frac{1}{N}\sum_{j=1}^{N} K(x, z_j(t))$$

$$d_t\phi(t, z_i(t)) = \partial_t\phi(t, z_i(t)) + \sum_{k=1}^{N} z_i'(t)_k \partial_k\phi(t, z_i(t))$$

$$\frac{1}{N}\sum_{j=1}^{N} K(z_i(t), z_j(t))_k = z_i'(t)_k$$

We calculate

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \partial_t \phi(t,x) + \mathcal{K}(\mu(t))(x) \cdot \nabla_x \phi(t,x) d\mu(t)(x) dt \\
&= \int_0^T \frac{1}{N} \sum_{i=1}^N [\partial_t \phi(t, z_i(t)) + \sum_{k=1}^N \frac{1}{N} \sum_{j=1}^N K(z_i(t), z_j(t))_k \partial_k \phi(t, z_i(t))] dt \\
&= \int_0^T \frac{1}{N} \sum_{i=1}^N [\partial_t \phi(t, z_i(t)) + \sum_{k=1}^N z_i'(t)_k \partial_k \phi(t, z_i(t))] dt \\
&= \frac{1}{N} \sum_{i=1}^N \int_0^T \partial_t \phi(t, z_i(t)) + \sum_{k=1}^N z_i'(t)_k \partial_k \phi(t, z_i(t)) dt \\
&= \frac{1}{N} \sum_{i=1}^N \int_0^T d_t \phi(t, z_i(t)) dt = 0
\end{aligned}
$$

Hence $\mu(t)$ is a weak solution of $\partial_t \mu + div_x(\mu \mathcal{K}(\mu)) = 0$ with initial condition $\mu(0) = \mu_Z$.

Theorem 3.2.2

Let $\mu^{in} \in \mathcal{P}_1(\mathbb{R}^d)$ be an initial measure.
Then there exists a unique solution $Z \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ of the system

$$\begin{cases} \partial_t Z(t,\zeta) = \mathcal{K}(\mu(t))(Z(t,\zeta)) \\ \mu(t) = Z(t,\cdot)\#\mu^{in} \\ Z(0,\zeta) = \zeta \end{cases}$$

such that for any $\zeta \in \mathbb{R}$ the map

$$Z(\cdot,\zeta) : \mathbb{R} \to \mathbb{R}^d; t \mapsto Z(t,\zeta)$$

lies $C^1(\mathbb{R},\mathbb{R}^d)$.

Proof

Set $C := \int\limits_{\mathbb{R}^d} |x|\, d\mu^{in}(x)$ and let

$$V := \left\{ v \in C(\mathbb{R}^d, \mathbb{R}^d) \middle| \sup_{x \in \mathbb{R}^d} \frac{|v(x)|}{1+|x|} < \infty \right\}$$

where $\|v\|_V := \sup\limits_{x \in \mathbb{R}^d} \frac{|v(x)|}{1+|x|}$. V is a Banach space with this norm.

$$\left| \int\limits_{\mathbb{R}^d} K(v(x), v(y)) d\mu^{in}(y) - \int\limits_{\mathbb{R}^d} K(w(x), w(y)) d\mu^{in}(y) \right|$$

$$\leqslant \int\limits_{\mathbb{R}^d} |K(v(x), v(y)) - K(v(x), w(y))| + |K(v(x), w(y)) - K(w(x), w(y))|\, d\mu^{in}(y)$$

$$\leqslant L \int\limits_{\mathbb{R}^d} |v(y) - w(y)| + |v(x) - w(x)|\, d\mu^{in}(y)$$

$$\leqslant L \|v - w\|_V (1 + |x|) + L \|v - w\|_V \int\limits_{\mathbb{R}^d} (1 + |y|) d\mu^{in}(y)$$

$$\leqslant L \|v - w\|_V (1 + |x| + 1 + C) \leqslant L \|v - w\|_V (1 + |x|)(2 + C)$$

Inductive Definition

$$\begin{cases} Z_{n+1}(t,\zeta) = \zeta + \int_0^t \int_{\mathbb{R}^d} K(Z_n(s,\zeta), Z_n(s,y))\,d\mu^{in}(y)\,ds \\ Z_0(t,\zeta) = \zeta \end{cases}$$

Then we have for $\zeta \in \mathbb{R}^d$ and $t \in \mathbb{R}$ that

$$\begin{aligned} |Z_1(t,\zeta) - Z_0(t,\zeta)| &\leqslant \left| \zeta + \int_0^t \int_{\mathbb{R}^d} K(Z_0(s,\zeta), Z_0(s,y))\,d\mu^{in}(y)\,ds - \zeta \right| \\ &\leqslant \left| \int_0^t \int_{\mathbb{R}^d} K(\zeta, y)\,d\mu^{in}(y)\,ds \right| \\ &\leqslant \left| \int_0^t \int_{\mathbb{R}^d} L(|\zeta| + |y| + |K(0)|)\,d\mu^{in}(y)\,ds \right| \\ &\leqslant |t|\, L(|\zeta| + C) \leqslant L\,|t|\,(2 + C)(1 + |\zeta|) \end{aligned}$$

Hence

$$\|Z_1(t,\cdot) - Z_0(t,\cdot)\|_V \leqslant L\,|t|\,(2 + C)$$

Inductively we see that

$$\|Z_{n+1}(t,\cdot) - Z_n(t,\cdot)\|_V \leqslant \frac{((2+C)L\,|t|)^{n+1}}{n!}$$

Indeed

$$\begin{aligned}
|Z_{n+2}(t,\zeta) - Z_{n+1}(t,\zeta)| &\leqslant \left| \int_0^t \int_{\mathbb{R}^d} K(Z_{n+1}(s,\zeta), Z_{n+1}(s,y)) - K(Z_n(s,\zeta), Z_n(s,y))\, d\mu^{in}(y)\, ds \right| \\
&\leqslant \left| \int_0^t \|Z_{n+1}(s,\cdot) - Z_n(s,\cdot)\|_V\, (2+C)(1+|\zeta|)\, ds \right| \\
&\leqslant \left| \int_0^t \frac{((2+C)L)^{n+2}\,|s|^{n+1}}{n!}\, ds \right| (1+|\zeta|) \\
&= \frac{((2+C)L\,|t|)^{n+2}}{(n+1)!}(1+|\zeta|)
\end{aligned}$$

Existence of the limit

The previous slide showed that for any $t \in \mathbb{R}$ the sequence $(Z_n(t,\cdot))_{n\in\mathbb{Z}_{>0}}$ is Cauchy in $(V, \|\cdot\|_V)$. Let $Z(t,\cdot) \in V$ be its limit for $t \in \mathbb{R}$. By the explicit estimate we have of the difference between $Z_{n+k}(t,\cdot)$ and $Z_n(t,\cdot)$, we see that $Z_n(\cdot,\cdot) \underset{n\to\infty}{\to} Z(\cdot,\cdot)$ in V uniformly on $[-T, T]$ for any $T > 0$.
Hence we have for $t \in \mathbb{R}$ that

$$Z(t,\zeta) = \zeta + \int_0^t \int_{\mathbb{R}^d} K(Z(s,\zeta), Z(s,y))d\mu^{in}(y)ds$$

In particular,

$$Z(0,\zeta) = \zeta + \int_0^0 \int_{\mathbb{R}^d} K(Z(s,\zeta), Z(s,y))d\mu^{in}(y)ds = \zeta$$

Continuity of the limit

We want to show $Z : \mathbb{R} \to V; t \mapsto Z(t, \cdot)$ is continuous.
For $r, t \in [-T, T]$ we calculate

$$\begin{aligned}
|Z_{n+1}(r,x) - Z_{n+1}(t,x)| &\leqslant \left| \int_r^t \int_{\mathbb{R}^d} K(Z_n(s,x), Z_n(s,y)) d\mu^{in}(y) ds \right| \\
&\leqslant \left| \int_r^t \int_{\mathbb{R}^d} L\,|Z_n(s,x)| + L\,|Z_n(s,y)| + |K(0)|\, d\mu^{in}(y) ds \right| \\
&\leqslant \left| \int_r^t L(1+|x|)(2+C)\, \|Z_n(s,\cdot)\|_V \, ds \right| \\
&\leqslant L(1+|x|)(2+C) \left| \int_r^t \sum_{k=0}^{n-1} \frac{((2+C)L\,|s|)^{k+1}}{k!} ds \right| \\
&\leqslant (1+|x|)\,|t-r|\, L(2+C) \sum_{k=0}^{\infty} \frac{((2+C)L(1+|T|))^{k+1}}{k!}
\end{aligned}$$

Therefore $Z \in C(\mathbb{R}, V)$ and hence

$$\mathbb{R} \to \mathbb{R}^d; s \mapsto \int_{\mathbb{R}^d} K(Z(s,\zeta), Z(s,y)) d\mu^{in}(y)$$

is continuous. Moreover,

$$\begin{aligned}\partial_t Z(t,\zeta) &= \int_{\mathbb{R}^d} K(Z(t,\zeta), Z(t,y)) d\mu^{in}(y) \\ &= \int_{\mathbb{R}^d} K(Z(t,\zeta), y) dZ(t,\cdot)\#\mu^{in}(y) \\ &= \mathcal{K}(\mu(t))(Z(t,\zeta))\end{aligned}$$

and Z is indeed the desired solution of our initial value problem.

Uniqueness

Suppose $Y \in C(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ is another solution. Then

$$\partial_t Y(t,\zeta) = \mathcal{K}(\mu(t))(Y(t,\zeta)) = \int\limits_{\mathbb{R}^d} K(Y(t,\zeta), y) dY(t,\cdot)\#\mu^{in}(y)$$
$$= \int\limits_{\mathbb{R}^d} K(Y(t,\zeta), Y(t,y)) d\mu^{in}(y)$$

This shows $Y(t,\zeta) = \zeta + \int\limits_0^t \partial_s Y(s,\zeta) ds = \zeta + \int\limits_0^t \int\limits_{\mathbb{R}^d} K(Y(s,\zeta), Y(s,y)) d\mu^{in}(y) ds$ so

$$Y(t,\zeta) - Z(t,\zeta) = \int\limits_0^t \int\limits_{\mathbb{R}^d} K(Y(s,\zeta), Y(s,y)) - K(Z(s,\zeta), Z(s,y)) d\mu^{in}(y) ds$$

$$\|Y(t,\cdot) - Z(t,\cdot)\|_V \leqslant L(2+C) \left| \int\limits_0^t \|Y(s,\cdot) - Z(s,\cdot)\|_V \, ds \right|$$

By Gronwall $\|Y(t,\cdot) - Z(t,\cdot)\|_V = 0$ which implies $Y = Z$.

Proposition 3.2.3

Let $(Z_i)_{i\in[N]} \in (\mathbb{R}^d)^N$ be a tuple of initial points and let $\mu^{in} = \frac{1}{N}\sum_{i=1}^{N} \delta_{Z_i}$ be the associated initial empirical measure.
If $Z \in C(\mathbb{R} \times \mathbb{R}^d)$ is the solution from Theorem 3.2.2 and $(z_i)_{i\in[N]} \in C^1(\mathbb{R}, (\mathbb{R}^d)^N)$ is the solution from Theorem 3.2.1, then

$$\forall t \in \mathbb{R} : z_i(t) = Z(t, Z_i)$$

Proof

If $A \subseteq \mathbb{R}^d$ is a measurable set, then

$$\mu(t)(A) = Z(t,\cdot)\#\mu^{in}(A) = \mu^{in}(Z(t,\cdot)^{-1}(A))$$
$$= \frac{1}{N}\sum_{j=1}^{N} \delta_{Z_j}(Z(t,\cdot)^{-1}(A))$$
$$= \frac{1}{N}\sum_{j=1}^{N} \delta_{Z(t,Z_j)}(A)$$

Using this we calculate the differential

$$d_t Z(t, Z_i) = \partial_t Z(t, Z_i) = \mathcal{K}(\mu(t))(Z(t, Z_i))$$
$$= \frac{1}{N}\sum_{j=1}^{N} K(Z(t,Z_i), y)\delta_{Z(t,Z_j)}(y)$$
$$= \frac{1}{N}\sum_{i=1}^{N} K(Z(t,Z_i), Z(t,Z_j))$$

We conclude using uniqueness and $\forall i \in [N] : z_i(0) = Z_i = Z(0, Z_i)$

Lemma

For any $\varphi \in C_c^2(\mathbb{R}^2, \mathbb{R})$ *we have*

$$\int\limits_{\mathbb{R}^2} \log(|x|)\Delta\varphi(x)dx = 2\pi\varphi(0)$$

Hence $\Delta\frac{\log(x)}{2\pi} = \delta_0$ *and* $\frac{\log(x)}{2\pi}$ *is the fundamental solution of the Laplace operator in dimension* 2*.*

Proof

Let $\Phi : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2$ be polar coordinates and set $\phi := \varphi \circ \Phi$.

$$\int\limits_{\mathbb{R}^2} \log(|x|)\Delta\phi(x)dx = \int_0^\infty \int_0^{2\pi} \log(r)(\frac{1}{r}\partial_r(r\partial_r\phi) + \frac{1}{r^2}\partial_\theta^2\phi)rdrd\theta$$

$$= \int_0^{2\pi} r\log(r)\partial_r\phi(r,\theta)|_{r=0}^{\infty} - \int_0^\infty d_r(\log(r))r\partial_r\phi drd\theta$$

$$+ \int_0^\infty \frac{\log(r)}{r}\partial_\theta\phi(r,\theta)|_{\theta=0}^{2\pi}dr$$

$$= \int_0^{2\pi} 0 - \int_0^\infty \partial_r\phi drd\theta + \int_0^\infty \frac{\log(r)}{r}0dr$$

$$= -\int_0^{2\pi} -\varphi(0)d\theta = 2\pi\varphi(0)$$

Thank you for your attention.