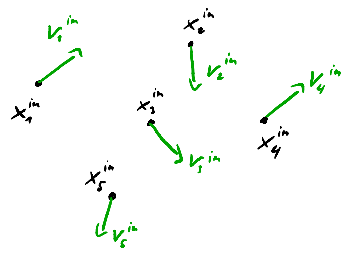
12.04.2021

Seminar in kinetic theory

Dobrushin's stability estimate and the mean field limit

1. Recap of the context

<u>t=0</u> $Z_N^{in} = (z_1^{in}, \dots, z_N^{in})$



$z_i^{in} = (x_i^{in}, v_i^{in})$

<u>t > 0</u> ?

$$(1)\quad \begin{cases} \dot{z}_i(t) = \dfrac{1}{N} \displaystyle\sum_{\substack{j=1 \\ j\neq 1}}^{N} k(z_i(t), z_j(t)) \\ z_i(0) = z_i^{in} \end{cases} \qquad i = 1, \dots, N$$

C^1 with bdd derivatives

good assumptions on k (Hk1 – Hk2)

↳ by Cauchy-Lipschitz $\exists!$ sol of (1)

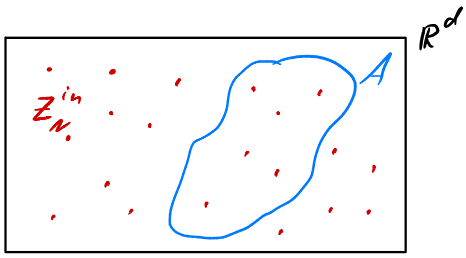
$$Z_N(t) = (z_1(t), \dots, z_N(t))$$

In real life, example of a galaxy, $N = 10^{11}$, plasma $N = 10^{23}$

⇝ not possible to solve numerically the system (1)

<u>Idea</u>: try to solve the discrete system of particules by a continous system

First introduce the empiricale measure of Z_N^{in} $\quad \mu_{Z_N^{in}} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \delta_{z_j^{in}}$



$A \subset \mathbb{R}^d$

$\mu_{Z_N^{in}}(A) = \#$ of particules in A

We know that $\mu(t) := \mu_{Z_N(t)}$ is a weak solution of the mean field PDE

$$(2)\quad \begin{cases} \partial_t \mu + \operatorname{div}_z (\mu \mathcal{K}(\mu)) = 0 \\ \mu(0) = \mu_{in} = \mu_{Z_N^{in}} \end{cases} \qquad \mathcal{K}(\mu(t))(z) = \int_{\mathbb{R}^d} k(z, z')\, \mu(t, dz')$$

<u>Remark</u> $Z_N(t) \in C^1(\mathbb{R}^{dN})$, i.e. the space where lives the solution depends on N but $\mu(t) \in \mathcal{P}(\mathbb{R}^d)$ does not depend on N.

The characteristic curves of this mean field PDE is given by

$$(3)\quad \begin{cases} \dot{Z}(t) = \mathcal{K}(\mu(t))(Z(t)) = \int_{\mathbb{R}^d} k(Z(t), z')\,\mu(t, dz') \\ Z(0) = z \end{cases}$$

where $\mu(t) = Z(t) \# \mu_{in}$ existence and uniquness (Thm 3.2.2)

if we start with the initial condition $Z_N^{in} = (z_1^{in}, \dots, z_N^{in})$ and consider $\mu_{Z_N^{in}}$ and the solution of N-ODE system $Z_N(t) = (z_1(t), \dots, z_N(t))$ then we have

$z_i(t) = Z(t)$ where $Z(t)$ solves $\begin{cases} \dot{Z}(t) = \mathcal{K}(\mu(t))(Z(t)) \\ Z(0) = z_i^{in} \end{cases}$

Prop 3.2.3

$\mu(t) = Z(t) \# \mu_{Z_N^{in}}$

<u>Remark</u> The mean field characteristic flow, $Z(t)$, contains all the relevant information about both the mean field PDE (2) and the N-particle ODE system (1).

<u>Sketch of what we are going to show</u>

<u>discrete system</u>

$$\begin{cases} (ODE)_N \\ (data)_N \end{cases} \rightsquigarrow (sol)_N$$

<u>continous system</u>

$$\begin{cases} (PDE) \\ (data) \end{cases} \rightsquigarrow (sol)$$

<u>Question</u>: $(data)_N \longrightarrow (data) \overset{?}{\Longrightarrow} (sol)_N \longrightarrow (sol)$

yes, using Dobrushin's estimate

But first we need some tools about distance between probability measures

2. The Monge-Kantorovich distance

$$\mathcal{P}_1(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} |z| \, \mu(dz) < \infty \right\}$$

$\hookrightarrow$ set of Borel probability measure on $\mathbb{R}^d$

mu nu

Given $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ we define

$$\Pi(\mu,\nu) = \left\{ \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \;\middle|\; \pi(A \times \mathbb{R}^d) = \mu(A) \text{ and } \pi(\mathbb{R}^d \times A) = \nu(A) \;\; \forall \text{ Borel set } A \right\}$$

set of Borel probability measure π on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second ν

Π is sometimes refered to as "couplings of μ and ν"

<u>Coupling property</u> $\displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x) + \psi(y) \;\; \pi(dx,dy) = \int_{\mathbb{R}^d} \phi(x)\,\mu(dx) + \int_{\mathbb{R}^d} \psi(y)\,\mu(dy)$

<u>Def</u> Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, the Monge-Kantorovic distance (or Wasserstein distance) between μ and ν is

$$W_1(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y| \;\; \pi(dx,dy)$$

<u>Def</u> Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, the bounded Lipschitz distance between μ and ν is

$$d_{bL}(\mu,\nu) := \sup_{\substack{\phi \in Lip(\mathbb{R}^d) \\ Lip(\phi) \le 1}} \left| \int_{\mathbb{R}^d} \phi(z) \;\mu(dz) - \int_{\mathbb{R}^d} \phi(z) \;\nu(dz) \right|$$

$$Lip(\phi) := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|}$$

possible to show that W_1 and d_{bL} are well defined distances

<u>Prop</u> Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, then $W_1(\mu,\nu) = d_{bL}(\mu,\nu)$ duality argument

<u>Def</u> A sequence $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}_1(\mathbb{R}^d)$ converges weakly to $\mu \in \mathcal{P}_1(\mathbb{R}^d)$ if $\forall \phi \in C_b^0(\mathbb{R}^d)$ we have

cont. and bdd

$$\int_{\mathbb{R}^d} \phi(z) \;\mu_n(dz) \xrightarrow[n \to \infty]{} \int_{\mathbb{R}} \phi(z)\,\mu(dz)$$

And we write $\mu_n \longrightarrow \mu$

<u>Prop</u> W_1 (or d_{bL}) induces the weak topology on $\mathcal{P}_1(\mathbb{R}^d)$. i.e.

$$W_1(\mu_n,\mu) \xrightarrow[n \to \infty]{} 0 \iff \mu_n \longrightarrow \mu$$

3. Dobrushin's estimate

We are given two initial measure $\mu_1^{in}, \mu_2^{in} \in \mathcal{P}_1(\mathbb{R}^d)$ doesn't matter if μ_{in} is cont. or not for now

$$(4)\begin{cases} \partial_t \mu_1 + \mathrm{div}_z(\mu_1 \mathcal{K}(\mu_1)) = 0 \\ \mu_1(0) = \mu_1^{in} \end{cases} \qquad (5)\begin{cases} \partial_t \mu_2 + \mathrm{div}_z(\mu_2 \mathcal{K}(\mu_2)) = 0 \\ \mu_2(0) = \mu_2^{in} \end{cases}$$

flow, $z_1 \in \mathbb{R}^d$ flow $z_2 \in \mathbb{R}^d$

$$\begin{cases} \dot{Z}_1(t) = \int_{\mathbb{R}^d} k(Z_1(t), z')\,\mu_1(t, dz') \\ Z_1(0) = z_1 \end{cases} \qquad \begin{cases} \dot{Z}_2(t) = \int_{\mathbb{R}^d} k(Z_2(t), z')\,\mu_2(t, dz') \\ Z_2(0) = z_2 \end{cases}$$

sol. sol

$$Z_1(t) = z_1 + \int_0^t \int_{\mathbb{R}^d} k(Z_1(s), z')\,\mu_1(s, dz')\,ds$$

$$Z_2(t) = z_2 + \int_0^t \int_{\mathbb{R}^d} k(Z_2(s), z')\,\mu(s, dz')\,ds$$

Now we have two well defined flow $Z_1(t)$, $Z_2(t)$ where $Z_i(t)$ depends on the initial distribution μ_i^{in} and the initial position in the phase space z_i. And the goal is to study the stability of these flows in order to prove the mean field limit.

Most computational part of my talk

$$Z_1(t) - Z_2(t) = z_1 - z_2 + \int_0^t \int_{\mathbb{R}^d} k(Z_1(s), z')\,\mu_1(s, dz')\,ds - \int_0^t \int_{\mathbb{R}^d} k(Z_2(s), z')\,\mu_2(s, dz')\,ds$$

$\mu_i(s) = Z_i(s) \# \mu_i^{in}$

$$= z_1 - z_2 + \int_0^t \int_{\mathbb{R}^d} k(Z_1(s), Z_1'(s))\,\mu_1^{in}(dz_1') - \int_{\mathbb{R}^d} k(Z_2(s), Z_2'(s))\,\mu_2^{in}(dz_2')\,ds$$

Coupling property $\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})$

$$= z_1 - z_2 + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} k(Z_1(s), Z_1'(s)) - k(Z_2(s), Z_2'(s))\ \pi^{in}(dz_1', dz_2')\,ds$$

$$\Rightarrow \quad |Z_1(t) - Z_2(t)|$$

$$\leq |z_1 - z_2| + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| k(Z_1(s), Z_1'(s)) - k(Z_2(s), Z_2'(s)) \right| \pi^{in}(dz_1', dz_2')\, ds$$

$$(Hk2) \quad |k(a,a') - k(b,b')| \leq L|a-b| + L|a'-b'|$$

$$\leq |z_1 - z_2| + L \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |Z_1(s) - Z_2(s)|\, \pi^{in}(dz_1', dz_2')\, ds$$

$$+ L \int_0^t \underbrace{\int_{\mathbb{R}^d \times \mathbb{R}^d} |Z_1'(s) - Z_2'(s)|\, \pi^{in}(dz_1', dz_2')}_{:= D[\pi^{in}](s)}\, ds$$

$$= |z_1 - z_2| + L \int_0^t |Z_1(s) - Z_2(s)|\, ds + L \int_0^t D[\pi^{in}](s)\, ds$$

$\Rightarrow$ *integrating both sides with respect to* $\pi^{in}(dz_1, dz_2)$

$$\underbrace{\int_{\mathbb{R}^d \times \mathbb{R}^d} |Z_1(t) - Z_2(t)|\, \pi^{in}(dz_1, dz_2)}_{= D[\pi^{in}](t)} \leq \underbrace{\int_{\mathbb{R}^d \times \mathbb{R}^d} |z_1 - z_2|\, \pi^{in}(dz_1, dz_2)}_{D[\pi^{in}](0)}$$

$$+ L \int_0^t \underbrace{\int_{\mathbb{R}^d \times \mathbb{R}^d} |Z_1(s) - Z_2(s)|\, \pi^{in}(dz_1, dz_2)}_{= D[\pi^{in}](s)}\, ds + L \int_0^t D[\pi^{in}](s)\, ds$$

$$\Rightarrow \quad D[\pi^{in}](t) \leq D[\pi^{in}](0) + 2L \int_0^t D[\pi^{in}](s)\, ds$$

Therefore by Grönwall's inequality, we have $\forall t \in \mathbb{R}$

$$D[\pi^{in}](t) \leq D[\pi^{in}](0)\, e^{2L|t|} \qquad (*)$$

Theorem (Dobrushin's stability Theorem)

Assume $k \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ satisfies (HK1 - HK2).

Let $\mu_1^{in}, \mu_2^{in} \in \mathcal{P}_1(\mathbb{R}^d)$. For all $t \in \mathbb{R}$, let

$\mu_1(t) = Z_1(t) \# \mu_1^{in}$ $\rightsquigarrow$ μ_1 sol. of (4), Z_1 characteristic flow

$\mu_2(t) = Z_2(t) \# \mu_2^{in}$ $\rightsquigarrow$ μ_2 sol of (5), Z_2 characteristic flow

Then for all $t \in \mathbb{R}$, we have

$$W_1(\mu_1(t), \mu_2(t)) \leq e^{2L|t|} W_1(\mu_1^{in}, \mu_2^{in})$$

Proof

$$W_1(\mu_1(t), \mu_2(t)) \overset{def.}{=} \inf_{\pi(t) \in \Pi(\mu_1(t), \mu_2(t))} \int_{\mathbb{R}^d \times \mathbb{R}^d} |z_1 - z_2| \, \pi(t, dz_1, dz_2)$$

Coupling + flow

$\pi(t) = (Z_1(t), Z_2(t)) \# \pi^{in}$

$$= \inf_{\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})} \int_{\mathbb{R}^d \times \mathbb{R}^d} |Z_1(t) - Z_2(t)| \, \pi^{in}(dz_1, dz_2)$$

$$= \inf_{\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})} D[\pi^{in}](t)$$

by (*)

$$\leq \inf_{\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})} D[\pi^{in}](0) \, e^{2L|t|}$$

$$= e^{2L|t|} W_1(\mu_1^{in}, \mu_2^{in})$$

$\square$

4. The mean field limit

This a consequence of Dobrushin's stability Theorem.

Theorem interaction kernel between the particles

Assume $k \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ satisfies (HK1 – HK2).

Let $f^{in} \in \mathcal{P}_1(\mathbb{R}^d)$ be a probability density on $\mathbb{R}^d$.

1) Then the Cauchy problem for the mean field PDE

$$\begin{cases} \partial_t f(t,z) + \operatorname{div}_z \big(f(t,z) \; \mathcal{K}(f(t,z)) \big) = 0 \\ f(0,z) = f^{in}(z) \end{cases}$$

has a unique weak solution $f \in C(\mathbb{R}, L^1(\mathbb{R}^d))$

2) For each $N \geq 1$, let $Z_N^{in} = (z_1^{in}, \dots, z_N^{in}) \in \mathbb{R}^{dN}$.

Let $Z_N(t) = (z_1(t), \dots, z_N(t))$ be the solution of the N-particles ODE with initial condition Z_N^{in}

$$\begin{cases} \dot{z}_i(t) = \dfrac{1}{N} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N} k(z_i(t), z_j(t)) \\ z_i(0) = z_i^{in} \end{cases}$$

and consider the empirical measure $\mu_{Z_N^{in}} = \frac{1}{N} \sum_{j=1}^{N} \delta_{z_j^{in}}$.

Assume that $\mu_{Z_N^{in}} \longrightarrow f^{in}$ as $N \to \infty$

Then $\mu_{Z_N(t)} \longrightarrow f(t,\cdot)$ as $N \to \infty$

"Proof" 1) $f(t) = Z(t) \# f^{in}$ Thm 3.2.1 and ex 6) 7) p.37

2) $\mu_{Z_N(t)} = Z_N(t) \# \mu_{Z_N^{in}}$ and $\mu_{Z_N^{in}} \xrightarrow[N \to \infty]{} f^{in} \iff W_1(\mu_{Z_N^{in}}, f^{in}) \xrightarrow[N \to \infty]{} 0$

Therefore by Dobrushin

$$W_1(f(t), \mu_{Z_N(t)}) \leq e^{2L|t|} W_1(\mu_{Z_N^{in}}, f^{in}) \xrightarrow[N \to \infty]{} 0$$

$$\Rightarrow \mu_{Z_N(t)} \xrightarrow[N \to \infty]{} f(t)$$

$\boxed{/\!/}$

In section 3.3.4. they show how to construct a sequence Z_N^{in} such that $\mu_{Z_N^{in}} \longrightarrow f^{in}$ as $N \to \infty$.

<u>Example</u> Vlasov - Poisson

$$z = (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad , \quad k(z, z') = \left(v - v' , C \frac{x - x'}{|x - x'|^3} \right)$$

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N} \sum\limits_{\substack{j=1 \\ j \neq i}} C \frac{x_i - x_j}{|x_i - x_j|^3} \end{cases}$$

$$\mu(t) = f(t)\, dx\, dv \quad , \quad \mathcal{K}(\mu(t)) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} k(x, v, x', v')\, f(t, x', v')\, dx'\, dv'$$

$$= \left(v , \nabla_x \phi(t, x) \right)$$

Mean field PDE $\partial_t \mu + \mathrm{div}_z \left(\mu \mathcal{K}(\mu) \right) = 0$ becomes

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi(t, x) \cdot \nabla_v f = 0 \\ -\Delta \phi(t, x) = C \int_{\mathbb{R}^3} f(t, x, v)\, dv \end{cases}$$

in this case $k(z, z') = \left(v - v' , C \frac{x - x'}{|x - x'|^3} \right) \notin C^1$

Still an open problem to justify correctly the mean field PDE in the case of Vlasov - Poisson !