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Seminar in kinetic theory

Vlasov - Poisson equation

1. How do we get this equation from the mean field PDE?

Recall the mean field PDE: Section 3.1 or Thm 3.3.4.

$$① \begin{cases} \partial_t f(t,z) + \operatorname{div}_z \left(f(t,z)\, \mathcal{K}(f(t,z)) \right) = 0 & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^d \\ f(0,z) = f^{in}(z) \end{cases}$$

where $\mathcal{K}(f(t,z)) = \int_{\mathbb{R}^d} k(z,z')\, f(t,z')\, dz'$, $k(z,z')$ is the interaction between two particles

Now in the case of Vlasov - Poisson we have

$z = (x,v) \in \mathbb{R}^d \times \mathbb{R}^d$ (we will mostly consider $d=2$ or $d=3$)

$$k(z,z') = k(x,v,x',v') = \left(v - v', \; C \frac{x-x'}{|x-x'|^d} \right)$$

$$\hookrightarrow \begin{cases} \dot{x} = v \\ \dot{v} = \Sigma F \end{cases}, \quad F(x,x') = C \frac{x-x'}{|x-x'|^d} \quad \text{is the Coulomb force}$$

Therefore we obtain:

$$\mathcal{K}(f(t,x,v)) = \left(v, -C \nabla_x \phi(t,x) \right)$$

with $-\Delta_x \phi(t,x) = c\, \rho(t,x) = c \int_{\mathbb{R}^d} f(t,x,v)\, dv$ (assume $C = 1$)

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} C \frac{x-x'}{|x-x'|^d} f(t,x',v')\, dx'dv' = \int_{\mathbb{R}^d} C \frac{x-x'}{|x-x'|^d} \underbrace{\int_{\mathbb{R}^d} f(t,x',v')\, dv'}_{= \rho(t,x')} dx'$$

$$-\nabla\left(\frac{1}{|x|^{d-2}}\right) = \frac{x}{|x|^d}$$

$$= -\nabla_x \int_{\mathbb{R}^d} C \frac{1}{|x-x'|^{d-2}} \rho(t,x')\, dx$$

$$= -C \nabla_x \phi(t,x)$$

Therefore

$$\operatorname{div}_{x,v}\big(f(t,x,v)\,\mathcal{H}(f(t,x,v))\big) = \operatorname{div}_x\big(f(t,x,v)\,v\big) + \operatorname{div}_v\big(f(t,x,v)\,(-\nabla_x\phi(t,x,v))\big)$$

$e : \mathbb{R}^n \to \mathbb{R}$, $\Psi : \mathbb{R}^n \to \mathbb{R}^n$

$\operatorname{div}(e\,\Psi) = \nabla e \cdot \Psi + e\,\operatorname{div}(\Psi)$

$$= v\cdot\nabla_x f(t,x,v) - \nabla_x\phi(t,x,v)\cdot\nabla_v f(t,x,v)$$

Therefore ① becomes the Vlasov - Poisson equation,

$$② \begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) - \nabla_x\phi(t,x)\cdot\nabla_v f(t,x,v) = 0 & (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d \\ -\Delta_x\phi(t,x) = \rho(t,x) \\ f(0,x,v) = f^{in}(x,v) \end{cases}$$

- $f(t,x,v)$ is the distribution function (of electron) at time t, position x, and velocity v.
- $\rho(t,x,v) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$ is the density.

ϕ is the potential

- $E(t,x) = -\nabla_x\phi(t,x)$ is the force field

"E is generated by the collective behaviour of all particles"

<u>Remark</u> ② is called the Vlasov - Poisson equation because the first line is a Vlasov type PDE and the second is the Poisson equation.
(The electric field in the Vlasov equation is given by the Poisson equation)

<u>Goal</u>: show that this system ② is well-posed. In the next weeks we will show the existence, (uniqueness) and regularity theory.

2. Elementary inequalities

Assumption on f: $\quad f \longrightarrow 0$ as $|x|, |v| \longrightarrow \infty$

Definition The mass and the energy of the V-P systeme are defined as follow :

$$\mathcal{M}(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v) \, dx \, dv$$

$$\mathcal{E}(t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2} |v|^2 f(t,x,v) \, dx \, dv + \int_{\mathbb{R}^d} \frac{1}{2} |\nabla_x \phi(t,x)|^2 \, dx$$

Proposition $\forall\, t \geq 0$ we have

$$\mathcal{M}(t) = \mathcal{M}(0) =: \mathcal{M}^{in}$$

$$\mathcal{E}(t) = \mathcal{E}(0) =: \mathcal{E}^{in}$$

Proof

$$\frac{d}{dt} \mathcal{M}(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \partial_t f(t,x,v) \, dx \, dv$$

V-P eq.

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} - v \cdot \nabla_x f(t,x,v) + \nabla_x \phi(t,x) \cdot \nabla_v f(t,x,v) \quad dx \, dv$$

$$= \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} - v \cdot \nabla_x f(t,x,v) \, dx}_{=0} \, dv + \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} \nabla_x \phi(t,x) \cdot \nabla_v f(t,x,v) \, dv}_{=0} \, dx$$

Integration by parts : $\nabla_x v = 0$ "boundary term $=0$" because $f \xrightarrow[|x| \to \infty]{} 0$

Integration by parts : $\nabla_v (\nabla_x \phi(t,x)) = 0$ "boundary term $=0$" because $f \xrightarrow[|v| \to \infty]{} 0$

$$= 0$$

$\Rightarrow \mathcal{M}(t) = \mathcal{M}(0)$ $\square$

Proposition $\forall\, p \in [1, \infty]$ $\quad \| f(t) \|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} = \| f^{in} \|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)}$

Positivity and maximum principle

If $0 \leq f^{in}(x,v) \leq M$ for a.e. $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$,

then we have $0 \leq f(t,x,v) \leq M$ for a.e. $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, $t \geq 0$

Proposition (Interpolation inequality) (not the same as $\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}$, $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$)

For each $0 \leq r \leq m$ $\exists\, C = C(d,r,m)$ s.t.

$$\left\| \int_{\mathbb{R}^d} |v|^r f(t,\cdot,v)\, dv \right\|_{L_x^{\frac{m+d}{r+d}}(\mathbb{R}^d)} \leq C \, \|f(t)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{m-r}{m+d}} \, \| |v|^m f(t) \|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{r+d}{m+d}}$$

Definition We define the moment of order r in the variable v as

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^r f(t,x,v)\, dx\, dv .$$

Example Let assume $f^{in} \geq 0$, $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, then

$$\|\rho(t)\|_{L_x^{\frac{m+d}{d}}(\mathbb{R}^d)} \leq C \Bigg(\underbrace{\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^m f(t,x,v)\, dx\, dv}_{\text{moments of order } m \text{ of the distribution}} \Bigg)^{\frac{d}{m+d}}$$

(apply the proposition with $r = 0$ and use $\|f(t)\|_{L^\infty} = \|f^{in}\|_{L^\infty} < C$)

Proof (idea)

$$\int_{\mathbb{R}^d} |v|^r f(t,x,v)\, dv = \int_{|v|<R} |v|^r f(t,x,v)\, dv + \int_{|v|>R} |v|^r f(t,x,v)\, dv$$

$$\leq \|f(t)\|_{L^\infty_{x,v}} R^r |B(0,R)| + \int_{|v|>R} \frac{|v|^{m-r}}{R^{m-r}} |v|^r f(t,x,v)\, dv$$

$$\leq \underbrace{\|f^{in}\|_{L^\infty_{x,v}} C}_{A} R^{r+d} + R^{r-m} \underbrace{\int_{\mathbb{R}^d} |v|^m f(t,x,v)\, dv}_{B}$$

we want : $A R^{r+d} = B R^{r-m}$ $\Rightarrow R = \left(\frac{B}{A}\right)^{\frac{1}{m+d}}$

$$= C \|f^{in}\|_{L^\infty_{x,v}}^{\frac{m-r}{m+d}} \left(\int_{\mathbb{R}^d} |v|^m f(t,x,v)\, dv \right)^{\frac{r+d}{m+d}}$$

by choosing $R = C \|f^{in}\|_{L^\infty_{x,v}}^{\frac{-1}{m+d}} \left(\int_{\mathbb{R}^d} |v|^m f(t,x,v)\, dx\, dv \right)^{\frac{1}{m+d}}$

and then take $L_x^{\frac{m+d}{r+d}}$ on both sides. $\boxtimes$

Estimates on the density

Proposition Let assume $f^{in} \geq 0$ a.e., $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathcal{E}^{in} := \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2} |v|^2 f^{in}(x,v)\, dx\, dv + \int_{\mathbb{R}^d} \frac{1}{2} |E^{in}(x)|^2\, dx < \infty$.

Then $\forall\ 1 \leq p \leq \frac{d+2}{d}$, $\rho(t) \in L^p(\mathbb{R}^d)$

i.e. $\exists\, C > 0$ s.t. $\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leq C$.

Proof We show $\rho(t) \in L^1(\mathbb{R}^d)$ and $\rho(t) \in L^{\frac{d+2}{d}}(\mathbb{R}^d)$

$\Rightarrow \rho(t) \in L^p(\mathbb{R}^d)$ $\forall\ 1 \leq p \leq \frac{d+2}{d}$ (Interpolation)

- $$\|\rho(t)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \rho(t,x)\, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(t,x,v)\, dv\, dx$$
$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} f^{in}(x,v)\, dx\, dv = \mathcal{M}^{in} < \infty$$

- $$\|\rho(t)\|_{L^{\frac{d+2}{d}}(\mathbb{R}^d)} \overset{\substack{\text{Interpolation inequality} \\ k=0,\ m=2}}{\leq} C\, \| |v|^2 f \|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{d}{d+2}} = C \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 f(t,x,v)\, dx\, dv \right)^{\frac{d}{d+2}}$$
$$\leq C \left(\mathcal{E}(t) \right)^{\frac{d}{d+2}} = C \left(\mathcal{E}^{in} \right)^{\frac{d}{d+2}} < \infty$$

Remark in dimension 2 and 3 we get $\rho(t) \in L^p(\mathbb{R}^d)$ for

$$p \in \begin{cases} [1, 2] & d = 2 \\ [1, \frac{5}{3}] & d = 3 \end{cases}$$

Estimates on the force field

$$E(t,x) = -\nabla_x \phi(t,x) \qquad , \quad -\Delta \phi(t,x) = \rho(t,x)$$

Definition We define the Green's function $G_d : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}$ for the operator $-\Delta$ as

$$G_d(x) = \begin{cases} -\frac{1}{2\pi} \ln(|x|) & d = 2 \\ \frac{1}{C(d)} \frac{1}{|x|^{d-2}} & d \geq 3 \end{cases} \qquad C(d) = (d-2)\,|B_1(0)|$$

G_d is also called the fundamental solution of the Laplace equation and G_d satisfies $\quad -\Delta G_d = \delta_0$

Furthermore, we know $\phi(t,x) = G_d * \rho(t,x)$ is the solution of the Poisson equation.

Hence $\quad E(t,x) = -\nabla \phi(t,x) = -\nabla G_d * \rho(t,x)$

$$= \int_{\mathbb{R}^d} C \frac{x-x'}{|x-x'|^d} \rho(t,x')\, dx'$$

$-\nabla G_d = C \frac{x}{|x|^d}$

Proposition Let $p, q > 0$ s.t. $1 + \frac{1}{q} = \frac{1}{p} + \frac{d-1}{d}$ $\quad \left(\Leftrightarrow \frac{1}{q} = \frac{1}{p} - \frac{1}{d} \,, \; 1 < p < d \right)$

Then $\|E(t)\|_{L^q(\mathbb{R}^d)} \leq C \, \|\rho(t)\|_{L^p(\mathbb{R}^d)}$

Proof Write $\quad \|E(t)\|_{L^p(\mathbb{R}^d)} = \|\nabla W * \rho(t)\|_{L^p(\mathbb{R}^d)}$

And use the weak Young inequality for convolution, namely

$\|f * g\|_{L^q} \leq C \, \|f\|_{L^{r,w}} \|g\|_{L^p}$ with $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, $1 < p, q, r < \infty$

and $\|\cdot\|_{L^{r,w}}$ is the weak L^p space, $\|f\|_{L^{r,w}} = \sup_{\delta > 0} \delta \, |\{x \in \mathbb{R}^d : |f(x)| > \delta\}|^{\frac{1}{p}}$

where $|\{\ldots\}|$ is the Lebesgue measure of the set $\{\ldots\}$.

And we show that $\nabla G_d \in L^{\frac{d}{d-1}, w}(\mathbb{R}^d)$.

$$\| \nabla G_d \|_{L^{\frac{d}{d-1},w}(\mathbb{R}^d)} \overset{def.}{=} \sup_{\gamma>0} \gamma \left| \left\{ x \in \mathbb{R}^d : |\nabla G_d| > \gamma \right\} \right|^{\frac{d-1}{d}}$$

$|\nabla G_d| = \left|\frac{x}{|x|^{d-2}}\right|$

$$= \sup_{\gamma>0} \gamma \left| \left\{ x \in \mathbb{R}^d : \left|\frac{1}{|x|^{d-1}}\right| > \gamma \right\} \right|^{\frac{d-1}{d}}$$

$$= \sup_{\gamma>0} \gamma \left| \left\{ x \in \mathbb{R}^d : |x| < \left(\frac{1}{\gamma}\right)^{\frac{1}{d-1}} \right\} \right|^{\frac{d-1}{d}}$$

ball of radius $\left(\frac{1}{\gamma}\right)^{\frac{1}{d-1}}$

$$= \sup_{\gamma>0} \gamma \left| B\left(0, \left(\frac{1}{\gamma}\right)^{\frac{1}{d-1}}\right) \right|^{\frac{d-1}{d}}$$

$|B(0,R)| = R^d \, |B(0,1)|$

$$= \sup_{\gamma>0} \gamma \left| \left(\frac{1}{\gamma}\right)^{\frac{d}{d-1}} B(0,1) \right|^{\frac{d-1}{d}}$$

$$= \sup_{\gamma>0} \gamma \cdot \frac{1}{\gamma} \, |B(0,1)|^{\frac{d-1}{d}} < C$$

$$\Rightarrow \quad \nabla G_d \in L^{\frac{d}{d-1},w}$$

$\boxed{}$

<u>Corollary</u> $\| E(t) \|_{L^q(\mathbb{R}^d)} \leq C$ for $\frac{d}{d-1} < q \leq \frac{d(d+2)}{(d-2)(d+1)}$.

i.e. $E \in L^q(\mathbb{R}^d)$ for $q \in \begin{cases} (2, \infty] & \text{if } d=2 \\ \left(\frac{3}{2}, \frac{15}{4}\right] & \text{if } d=3 \end{cases}$

We can also obtain estimates on $D_x E(t,x) = D^2 G_d * \rho(t,x)$

First we can compute

$$\left(D^2 G_d\right)_{ij} = \begin{cases} -\dfrac{x_i x_j}{|x|^{d+2}} & i \neq j \\[2ex] \dfrac{x_i^2 - \frac{1}{d}|x|^2}{|x|^{d+2}} & i = j \end{cases} \qquad \text{(modulo some constant)}$$

and we can show that $(D^2 G_d)_{ij}$ is a Calderón-Zygmund kernel and apply the C-Z. continuity Thm for singular integrals, namely

$$\| k * f \|_{L^p} \leq C \| f \|_{L^p} \quad \text{for } 1 < p < \infty$$

$$\Rightarrow \| D_x E(t) \|_{L^p} = \| D_x^2 G_d * \rho(t) \|_{L^p} \leq \| \rho(t) \|_{L^p} \quad \forall \; 1 < p < \infty$$

and $\| D_x E(t) \|_{L^p} \leq \| \rho(t) \|_{L^p} < C \quad \forall \; 1 < p \leq \frac{d+2}{d}$