The goal of this chapter is to prove the following theorem:

Theorem (Arsenev, 1975): Assume that $d \geq 2$ and let $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ satisfy $f^{in} \geq 0$ and $\mathcal{E}^{in} < \infty$. Then there exists a global weak solution $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v))$ of the Cauchy problem for the Vlasov-Poisson system with initial data f^{in}. Moreover, this solution satisfies

$$0 \leq f(t,x,v) \leq \|f^{in}\|_{L^\infty}$$

for a.e. $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$ and for all $t \geq 0$, together with mass bound

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v)\, d(x,v) \leq M^{in} < \infty$$

for all $t \geq 0$ and the energy bound

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|v|^2 f(t,x,v)\, d(x,v) + \int_{\mathbb{R}^d} \frac{1}{2}|\nabla_x \Phi(t,x)|^2\, dx \leq \mathcal{E}^{in} < \infty$$

for a.e. $t \geq 0$.

The initial condition is verified in the sense of distributions i.e. for all $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, the function

$$t \longmapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v)\varphi(x,v)\,d(x,v)$$

is continuous on $\mathbb{R}_+$ and satisfies

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(0,x,v)\varphi(x,v)\,d(x,v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f^{in}(x,v)\varphi(x,v)\,d(x,v).$$

Recall that the Cauchy problem for the Vlasov-Poisson system is

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) - \nabla_x \Phi(t,x) \cdot \nabla_v f(t,x,v) = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}^d, \\ -\Delta_x \Phi(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv, \\ f(0,x,v) = f^{in}(x,v). \end{cases}$$

To prove this theorem from Arsenev we first introduce the approximate Vlasov-Poisson system. This is done via mollifiers.

We introduce mollifiers as follows:
Let $\zeta \in C^\infty(\mathbb{R}^d)$ satisfy $\zeta(x) = \zeta(-x) \geq 0$ for all $x \in \mathbb{R}^d$, $\mathrm{supp}(\zeta) \subset B(0,1)$, and $\int_{\mathbb{R}^d} \zeta(x)\,dx = 1$, and set

$$\zeta_\epsilon(x) = \epsilon^{-d} \zeta(x/\epsilon),$$

$$\zeta_\epsilon(x,v) = \zeta_\epsilon(x)\,\zeta(v).$$

We use even mollifiers so that the convolution operator is self-adjoint on $L^2(\mathbb{R}^N)$.

Lemma:
Let $\chi \in C_c^\infty(\mathbb{R}^N)$ satisfy $\chi(x) = \chi(-x)$ for all $x \in \mathbb{R}^N$. Then the convolution operator $C_\chi : \phi \mapsto \chi * \phi$ is self-adjoint on $L^2(\mathbb{R}^N)$.

Proof:
First we show that C_χ is bounded. Let $\phi \in L^2(\mathbb{R}^N)$. Then

$$\| C_\chi \phi \|_{L^2} \leq \| \chi \|_{L^1} \| \phi \|_{L^2}$$

by Young's inequality for the convolution. Hence to show that it is self-adjoint it is sufficient to verify that it is symmetric.
Let $\psi \in L^2(\mathbb{R}^N)$. Then

$$\begin{aligned}(C_\chi \phi, \psi) &= (\chi * \phi, \psi) \\ &= (\phi, \chi * \psi).\end{aligned}$$

Indeed, by Fubini's theorem and since χ is even,

$$\int_{\mathbb{R}^N} \chi * \phi(x)\,\psi(x)\,dx$$
$$= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \chi(x-t)\,\phi(t)\,dt\right)\psi(x)\,dx$$
$$= \int_{\mathbb{R}^N} \phi(t)\left(\int_{\mathbb{R}^N} \chi(t-x)\,\psi(x)\,dx\right)dt$$
$$= \int_{\mathbb{R}^N} \phi(t)\,\chi * \psi(t)\,dt.$$

$\square$

We define the approx. Vlasov-Poisson system for a given $\epsilon > 0$ as

$$\begin{cases} \partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon - \nabla_x \Phi_\epsilon \cdot \nabla_v f_\epsilon = 0, \\ -\Delta_x \Phi_\epsilon(t,\cdot) = \zeta_\epsilon * \zeta_\epsilon * \rho_\epsilon(t,\cdot), \\ \nabla_x \Phi_\epsilon \to 0 \text{ as } |x|\to\infty, \\ \rho_\epsilon(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv, \\ f_\epsilon|_{t=0} = \zeta_\epsilon * (1_{\{\epsilon|x|<1\}} 1_{\{\epsilon|v|<1\}} f^{in}) =: f_\epsilon^{in}. \end{cases} \qquad (PV_\epsilon)$$

Thus for all $t \geq 0$,

$$-\nabla_x \Phi_\epsilon(t,\cdot) = (\zeta_\epsilon * \zeta_\epsilon * \nabla G_d) * \rho_\epsilon(t,\cdot)$$

where G_d is the d-dimensional Green's function, the fundamental solution.

For each $\epsilon > 0$, one has

$$\zeta_\epsilon * \zeta_\epsilon * \nabla G_d \in C^\infty(\mathbb{R}^d) \cap L^{\frac{d}{d-1},\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

Proposition:
For each $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ satisfying $f^{in} \geq 0$ and

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \tfrac{1}{2}|v|^2 f_\epsilon^{in}(x,v)\, d(x,v) + \int_{\mathbb{R}^d} \tfrac{1}{2} |\tilde{E}_\epsilon^{in}(x)|^2 dx = \mathcal{E}^{in} < \infty,$$

with

$$\tilde{E}_\epsilon^{in} = -\zeta_\epsilon * \nabla G_d * \rho_\epsilon^{in},$$

where

$$\rho_\epsilon^{in}(x) = \int_{\mathbb{R}^d} f_\epsilon^{in}(x,v)\, dv,$$

there exists a unique weak solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v))$ of (VP_ϵ).
This solution satisfies

$$0 \leq f_\epsilon(t,x,v) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}$$

for all $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$ and $t \geq 0$, together with mass conservation

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f_\epsilon(t,x,v)\, d(x,v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f_\epsilon^{in}(x,v)\, d(x,v)$$

and the approximate energy conservation bound

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\tfrac{1}{2}|v|^2 f_\epsilon(t,x,v)\,d(x,v)+\int_{\mathbb{R}^d}\tfrac{1}{2}|\tilde{E}_\epsilon(t,x)|^2dx$$
$$\leq \mathcal{E}^{in}$$

for all $t\geq 0$, where

$$\tilde{E}_\epsilon(t,\cdot)=(\zeta_\epsilon * \nabla G_d) * \rho_\epsilon(t,\cdot).$$

Proof:
Wlog, we may assume that $\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon^{in}(x,v)\,d(x,v)=1$ and that $\int_{\mathbb{R}^d\times\mathbb{R}^d} v f_\epsilon^{in}(x,v)\,d(x,v)=0$. Since the dynamics of the approximate Vlasov-Poisson system preserves the total momentum, total mass and total momentum, one has

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon(t,x,v)\,d(x,v)=1 \quad \text{for all } t\geq 0,$$

and

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} v f_\epsilon(t,x,v)\,d(x,v)=0 \quad \text{for all } t\geq 0.$$

We have already seen that the Vlasov-Poisson system can be put in the form

$$\partial_t f_\epsilon(t,z)+\operatorname{div}_z\Big(f_\epsilon(t,z)\int_{\mathbb{R}^{2d}} K_\epsilon(z,z')f_\epsilon(t,z')dz'\Big)=0,$$

with $z=(x,v)$ and

$$K_\epsilon(x,v,x',v') = (v-v', \zeta_\epsilon * \zeta_\epsilon * \nabla G_d).$$

We have seen that in this formalism, the existence and uniqueness is obtained via the construction of the mean field flow. Indeed, there exists a unique map

$$\mathbb{R}_+ \times \mathbb{R}^{2d} \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^{2d}$$
$$(t, z^{in}, \mu^{in}) \longmapsto Z_\epsilon(t, z^{in}, \mu^{in})$$

such that $t \mapsto Z_\epsilon(t, z^{in}, \mu^{in})$ is the integral curve of the vector field

$$z \longmapsto \int_{\mathbb{R}^{2d}} K_\epsilon(z,z')\mu_\epsilon(t, dz')$$
$$:= (K_\epsilon \mu_\epsilon(t))(z)$$

passing through z^{in} at time $t=0$, where $\mu_\epsilon(t) := Z_\epsilon(t, \cdot, \mu^{in}) \# \mu^{in}$.
Set

$$Z_\epsilon(t, z^{in}) := Z_\epsilon(t, z^{in}, f_\epsilon^{in} \mathcal{L}^{2d})$$

and

$$V_\epsilon(t,z) := (K_\epsilon \mu_\epsilon(t))(z)$$

where $\mu_\epsilon(t) := Z_\epsilon(t, \cdot) \# (f_\epsilon^{in} \mathcal{L}^{2d})$.

We have that

$$V_\epsilon(t,z) = (v-v', -(\zeta_\epsilon * \zeta_\epsilon * \nabla G_d) * \rho_\epsilon).$$

Hence, $V_\epsilon \in C(\mathbb{R}_+ \times \mathbb{R}^{2d}; \mathbb{R}^{2d})$ and $V_\epsilon(t,\cdot) \in C^\infty(\mathbb{R}^{2d}; \mathbb{R}^{2d})$. Moreover, we have an estimate on $V_\epsilon(t,z)$. By Young's inequality, we get

$$|V_\epsilon(t,z)| \le |v-v'| + \|\rho_\epsilon\|_{L^1} \|\zeta_\epsilon * \zeta_\epsilon * \nabla G_d\|_{L^\infty}$$

$$\le |z| + \|\zeta_\epsilon\|_{L^1} \|\zeta_\epsilon * \nabla G_d\|_{L^\infty}$$

$$\le |z| + \|\zeta_\epsilon * \nabla G_d\|_{L^\infty}.$$

Hence $Z_\epsilon \in C^1(\mathbb{R}_+ \times \mathbb{R}^{2d}; \mathbb{R}^{2d})$ and $Z_\epsilon(t,\cdot)$ is a C^∞-diffeomorphism of $\mathbb{R}^{2d}$ for all $t \ge 0$. Since $V_\epsilon(t,\cdot)$ is divergence free, the diffeomorphism $Z_\epsilon(t,\cdot)$ leaves the Lebesgue measure $\mathcal{L}^{2d}$ invariant. Therefore, the solution of (VP_ϵ) is of the form

$$\begin{aligned}
\mu(t) &= Z_\epsilon(t,\cdot) \# (\rho_\epsilon^{in} \mathcal{L}^{2d}) \\
&= \rho_\epsilon^{in}(Z_\epsilon(t,\cdot)^{-1}) \mathcal{L}^{2d} \\
&:= \rho_\epsilon(t,\cdot) \mathcal{L}^{2d}
\end{aligned}$$

for all $t \ge 0$.

Then mass conservation is satisfied since $Z_\epsilon(t,\cdot)$ is a diffeomorphism leaving invariant the Lebesgue measure. Continuity is obvious. We have to check the energy conservation. Since $f_\epsilon^{in} \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we have that $f_\epsilon \in C^1(\mathbb{R}_+ \times \mathbb{R}^{2d})$ and $\mathrm{supp}(f_\epsilon(t,\cdot))$ is compact for all $t \geq 0$. By Lebesgue's dominated convergence theorem we get

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} \frac{1}{2}|v|^2 f_\epsilon(t,x,v)\, d(x,v)$$

$$= \int_{\mathbb{R}^{2d}} \frac{1}{2}|v|^2 \partial_t f_\epsilon(t,x,v)\, d(x,v)$$

$$= \int_{\mathbb{R}^{2d}} \frac{1}{2}|v|^2 \big(\nabla_x \Phi_\epsilon(t,x) \cdot \nabla_v f_\epsilon(t,x,v) - v \cdot \nabla_x f_\epsilon(t,x,v)\big)\, d(x,v)$$

$$= \int_{\mathbb{R}^{2d}} -\mathrm{div}_x\Big(\frac{1}{2}|v|^2 v f_\epsilon(t,x,v)\Big)\, d(x,v) + \int_{\mathbb{R}^{2d}} \frac{1}{2}|v|^2 \nabla_v f_\epsilon(t,x,v) \cdot \nabla_x \Phi_\epsilon(t,x)\, d(x,v).$$

The first term vanishes using the divergence theorem and the fact that f_ε^{in} has compact support. By integration by parts, we get

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^{2d}} \tfrac{1}{2}|v|^2 f_\varepsilon(t,x,v)\,d(x,v)
&= -\int_{\mathbb{R}^{2d}} v\cdot\nabla_x \Phi_\varepsilon(t,x)\, f_\varepsilon(t,x,v)\,d(x,v)\\
&= -\int_{\mathbb{R}^{d}} j_\varepsilon(t,x)\cdot\nabla_x \Phi_\varepsilon(t,x)\,dx\\
&= \int_{\mathbb{R}^{d}} j_\varepsilon(t,x)\cdot\left(\zeta_\varepsilon * \tilde{E}_\varepsilon(t,\cdot)\right)dx\\
&= \int_{\mathbb{R}^{2d}} \left(\zeta_\varepsilon * j_\varepsilon(t,\cdot)\right)(x)\cdot \tilde{E}_\varepsilon(t,x)\,dx
\end{aligned}
$$

for all $t>0$ where we use the fact that the convolution operator with even function is self-adjoint for the last equality.

Then, using Green's identity, we get

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} \frac{1}{2}|v|^2 f_t(y,x,v)\, d(x,v)$$

$$= \int_{\mathbb{R}^d} (\varphi_\epsilon * \operatorname{div}_x j_t(y,\cdot))(x)\, \tilde{\Phi}_t(y,x)\, dx$$

$$= -\int_{\mathbb{R}^d} (\varphi_\epsilon * \partial_t \rho_t(y,\cdot))(x)\, \tilde{\Phi}_t(y,x)\, dx$$

$$= \int_{\mathbb{R}^d} (\partial_t \Delta_x \tilde{\Phi}_t(y,x))\, \tilde{\Phi}_t(y,x)\, dx$$

$$= -\int_{\mathbb{R}^d} \partial_t \nabla_x \tilde{\Phi}_t(y,x) \cdot \nabla_x \tilde{\Phi}_t(y,x)\, dx$$

$$= -\frac{d}{dt}\int_{\mathbb{R}^d} \frac{1}{2}|\nabla_x \tilde{\Phi}_t(y,x)|^2\, dx$$

for all $t \geq 0$, and hence the energy is conserved.

$\square$