4.2.2 Global Existence of Weak Solutions (Convergence to the Vlassov-Poisson System)

In this section we're going to prove the global existence of weak solutions of the Cauchy Problem for the Vlasov-Poisson system:

$(E^{in}(x) = -\nabla G_d * \rho^{in}(x),$
$\rho^{in}(x) = \int_{\mathbb{R}^d} f^{in}(x,v)dv$

<u>Thm 4.2.1:</u> Let $d \geq 2$ and $f^{in} \in L^1 \cap L^\infty(\mathbb{R}_x^d \times \mathbb{R}_v^d)$ satisfy

$$f^{in} \geq 0 \text{ a.e. and } \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{2}|v|^2 f^{in}(x,v)dxdv + \int_{\mathbb{R}^d} \frac{1}{2}|E^{in}(x)|^2 dx = \mathcal{E}^{in} < \infty.$$

Then there exists a global weak solution $f \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}_x^d\times\mathbb{R}_v^d))$ of the Cauchy problem for the Vlasov-Poisson system with initial data f^{in}. It satisfies

$$0 \leq f(t,x,v) \leq \|f^{in}\|_{L^\infty} \text{ for a.e. } (x,v) \in \mathbb{R}^d\times\mathbb{R}^d \text{ for all } t \geq 0,$$

together with the mass bound

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)dxdv \leq \mathcal{M}^{in} < \infty$$

for all $t \geq 0$ and the energy bound

$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)dv$
$E(t,x) = -\nabla_x G_d * \rho(t,x)$

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{2}|v|^2 f(t,x,v)dxdv + \int_{\mathbb{R}^d} \frac{1}{2}|E(t,x)|^2 dx \leq \mathcal{E}^{in} < \infty$$

for a.e. $t \geq 0$. The initial condition is verified in the sense of distributions, i.e. for all $\phi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, the function

$$t \longmapsto \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\phi(x,v)dxdv$$

is continous on $\mathbb{R}_+$ and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(0,x,v)\phi(x,v)dx\,dv = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\phi(x,v)dxdv$$

Proof: First, we recall the approximate Vlasov-Poisson system (VP_ε) for a $\varepsilon > 0$:

$$\begin{cases} \partial_t f_\varepsilon(t,x,v) + v \cdot \nabla_x f_\varepsilon(t,x,v) - \nabla_x \Phi_\varepsilon(t,x) \cdot \nabla_v f_\varepsilon(t,x,v) = 0, \\ -\Delta_x \Phi_\varepsilon(t,\cdot) = \zeta_\varepsilon * \zeta_\varepsilon * \rho_\varepsilon(t,\cdot), \qquad \nabla_x \Phi_\varepsilon \to 0 \text{ as } |x| \to \infty, \\ \rho_\varepsilon(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,dv, \\ f_\varepsilon|_{t=0} = \xi_\varepsilon * (\mathbb{1}_{\varepsilon|x|<1} \mathbb{1}_{\varepsilon|v|<1} f^{in}) =: f_\varepsilon^{in} \end{cases}$$

where $\zeta \in C^\infty(\mathbb{R}^d)$ is a even mollifier and $\zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(\frac{x}{\varepsilon})$ and $\xi_\varepsilon(x,v) := \zeta_\varepsilon(x)\zeta_\varepsilon(v)$. The Proof will follow 4 steps: we start by proving some uniform bound in ε for the families $\{f_\varepsilon\}$, $\{\rho_\varepsilon\}$ and $\{E_\varepsilon\}$. Then, we find candidates f, ρ, E for our solution via weak compactness and prove that they have most of the wanted properties. Then, we prove that f satisfies the Vlasov-Poisson PDE. Lastly, we show that the initial condition will be satisfied in a distributional sense.

step 1: consider $E_\varepsilon^{in} = -\zeta_\varepsilon * \nabla G_d * \rho_\varepsilon^{in}$ and $\rho_\varepsilon^{in}(x) = \int_{\mathbb{R}^d} f_\varepsilon^{in}(x,v)\,dv$ and $\frac{1}{q} = \frac{d}{d+2} - \frac{1}{d}$. With Youngs inequality (with $r=q$) and the interpolation inequality (with $r=0$ and $m=2$) ~~[illegible]~~ and the field estimate (where $\frac{1}{q} = \frac{1}{p} - \frac{1}{d} \Rightarrow p = \frac{d+2}{d}$) we get

$$\begin{aligned} \|E_\varepsilon^{in}\|_{L^q(\mathbb{R}^d)} &\leq \|\zeta_\varepsilon\|_{L^1(\mathbb{R}^d)} \|\nabla G_d * \rho_\varepsilon^{in}\|_{L^q(\mathbb{R}^d)} && \text{(Young)} \\ &\leq C_d \|\rho_\varepsilon^{in}\|_{L^{\frac{d+2}{d}}(\mathbb{R}^d)} && \text{(field est.)} \\ &\leq C_d(p,d) \|f_\varepsilon^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{2}{d+2}} \||v|^2 f_\varepsilon^{in}\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{d}{d+2}} && \text{(interp. ineq.)} \end{aligned}$$

$$\leq C_d\, C(p,d) \| f_\varepsilon^{in} \|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}^{\frac{2}{d+2}} (\mathcal{E}^{in})^{\frac{d}{d+2}} =: K$$

In the last inequality we used

$$\| |v|^2 f_\varepsilon^{in} \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = \| \xi_\varepsilon * (|v|^2 \mathbb{1}_{\varepsilon|x|<1} \mathbb{1}_{\varepsilon|v|<1} f^{in}) \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$$

$$\overset{(\text{Young})}{\leq} \| |v|^2 \mathbb{1}_{\varepsilon|x|<1} \mathbb{1}_{\varepsilon|v|<1} f^{in} \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$$

$$\leq \| |v|^2 f^{in} \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \leq \mathcal{E}^{in}$$

The last ineq. is by assumption. In a similar way we get

$$\| E_\varepsilon^{in} \|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \| \nabla G_d * \rho_\varepsilon^{in} \|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \quad (\text{Young})$$

$$\leq C_d \| \rho_\varepsilon^{in} \|_{L^1(\mathbb{R}^d)} \quad \left(\begin{array}{c}\text{field est.} \\ \text{with } p = 1 \text{ and } q = \frac{d-1}{d}\end{array}\right)$$

$$\leq C_d \| \rho^{in} \|_{L^1(\mathbb{R}^d)} = C_d M^{in}$$

The last inequality comes from the fact

$$M_\varepsilon^{in} = \| \rho_\varepsilon^{in} \|_{L^1(\mathbb{R}^d)} = \| f_\varepsilon^{in} \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = \| \xi_\varepsilon * (\mathbb{1}_{\varepsilon|x|<1} \mathbb{1}_{\varepsilon|v|<1} f^{in} \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$$

$$\overset{(\text{Young})}{\leq} \| \mathbb{1}_{\varepsilon|x|<1} \mathbb{1}_{\varepsilon|v|<1} f^{in} \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \leq \| f^{in} \|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$$

$$= \| \rho^{in} \|_{L^1(\mathbb{R}^d)} = M^{in}$$

This also shows the mass bound.

We know $q = \frac{d(d+2)}{(d-2)(d-1)}$ and let $p = \frac{d}{d-1}$. Let's assume for $r=2$ we have $p \leq r \leq q \iff 2 \leq d \leq 5$. There is a $\theta \in [0,1]$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. We know that

$$\|E_\varepsilon^{in}\|_{L^2(\mathbb{R}^d)} \leq \|E_\varepsilon^{in}\|_{L^p(\mathbb{R}^d)}^{\theta} \|E_\varepsilon^{in}\|_{L^q(\mathbb{R}^d)}^{1-\theta}$$

$$\leq (C_d M^{in})^\theta K^{1-\theta} < \infty$$

which is independent of $\varepsilon > 0$. Hence,

$$\sup_{\varepsilon > 0} \|E_\varepsilon^{in}\|_{L^2(\mathbb{R}^d)} < \infty$$

We also have shown that $\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|v|^2 f_\varepsilon^{in}(x,v)\,dv\,dx \leq \mathcal{E}^{in}$ is independent of $\varepsilon > 0$. Together we get with energy conservation that

$$\sup_{t,\varepsilon > 0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|v|^2 f_\varepsilon(t,x,v)\,dx\,dv + \int_{\mathbb{R}^d} \frac{1}{2}|E_\varepsilon(t,x)|^2 dx$$

$$= \sup_{\varepsilon > 0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|v|\, f_\varepsilon^{in}(x,v)\,dx\,dv + \int_{\mathbb{R}^d} \frac{1}{2}|E_\varepsilon^{in}(x)|^2 dx =: \mathcal{E}^{in} < \infty.$$

$$\Rightarrow \sup_{t,\varepsilon > 0} \|\tilde{E}_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^d)} < \infty$$

Step 2: By Banach-Alaoglu the unit ball in the dualspace is weak*-compact. Which implies that any bounded sequence has a convergent subsequence, provided the dualspace is metrizable, so that compactnes is equivalent to sequential-compactness. In our case we know that $L^\infty = (L^1)'$ for a σ-finite measure and the Lebesgue measure is. L^∞ is a separable Banach space, so the unit ball or any ball is metrizable. Since $(f_\varepsilon, E_\varepsilon)$ is uniformly bounded in $\varepsilon > 0$ (this & previous section) we can find a convergent subsequence of it (for the sake of simplicity we still call it $(f_\varepsilon, E_\varepsilon)$. Hence, we get

$$f_\varepsilon \to f \quad \text{in} \quad L^\infty(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d) \quad \text{weak-}*$$

$$\rho_\varepsilon \to \rho \quad \text{in} \quad L^\infty(\mathbb{R}_+ ; L^{\frac{d+2}{d}}(\mathbb{R}^d)) \quad \text{weak-}*$$

$$E_\varepsilon \to E \quad \text{in} \quad L^\infty(\mathbb{R}_+ ; L^2(\mathbb{R}^d)) \quad \text{weak-}*$$

(we have shown $\|\rho_\varepsilon(t)\|_{L^{\frac{d+2}{d}}(\mathbb{R}^d)}$, $\|E_\varepsilon(t)\|_{L^2(\mathbb{R}^d)} < \infty$ in section 4.1 and we have $\frac{d+1}{d-1} \leq 2 \leq \frac{d(d+2)}{(d-2)(d+1)}$)

Since we're working with the weak*-top., we're basicly working with distributions. So from now on, we're looking at distributions essentially. We have

$$0 = \partial x_i \partial x_j \Phi_\varepsilon - \partial x_j \partial x_i \Phi_\varepsilon = \partial x_j (E_\varepsilon)_i - \partial x_i (E_\varepsilon)_j \to \partial x_j E_i - \partial x_i E_j$$

in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^d)$ since

$$\int_{\mathbb{R}^d} \partial x_j (E_\varepsilon)_i(x) \phi(x)\, dx = - \int_{\mathbb{R}^d} (E_\varepsilon)_i(x) \partial x_j \phi(x)\, dx$$

$$= - \int_{\mathbb{R}^d} E_i(x) \partial x_j \phi(x)\, dx$$

$$= \int_{\mathbb{R}^d} \partial x_j E_i(x) \phi(x)\, dx$$

for every testfunction $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\partial_{x_j}\phi$ is also a testfunction. So there, must be a $\Phi \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^d)$ such that

$$E = -\nabla_x \Phi$$

by classical analysis. Since $\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{2}|v|^2 f_\varepsilon(t,x,v)\,dx\,dv \leq \mathcal{E}^{in} < \infty$

we see that the sequence f_ε is tight in the v variable (i.e. for a sequence μ_n of bounded, signed Radon measures is tight if $\mu_n(\mathbb{R}^n \setminus B(0,R)) \to 0$ as $R \to \infty$ uniformly in n). Using Fatou's Lemma we get

$$\int_{\mathbb{R}^d} \rho(t,x)\,dx \leq \liminf_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} \rho_\varepsilon(t,x)\,dx$$

$$= \liminf_{\varepsilon \searrow 0} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t,x,v)\,dv\,dx \leq M^{in}$$

So $\rho \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d))$. In the same way we get

$$\int_{\mathbb{R}^d} f(t,x,v)\,dv \leq \liminf_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,dv$$

$$= \liminf_{\varepsilon \searrow 0} \rho_\varepsilon(t,x) = \rho(t,x)$$

Using portmanteau's theorem, i.e. for a tight sequence of bounded, signed Radon measures μ_n, we have

$$\int \phi(z)\mu_n(dz) \to 0 \qquad \text{as } n \to \infty$$

for all $\phi \in C_b$ $\Leftrightarrow$ if it holds for all $\phi \in C_c$, we get equality.

to clarify $\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{2}|v|^2 f_\varepsilon(t,x,v)\,dx\,dv$ implies $f_\varepsilon(t,x,v) = o\left(\frac{1}{|v|^2}\right)$

which is independent $\forall\varepsilon$, so $\left(\mu_\varepsilon(\cdot) = \int_\cdot f_\varepsilon(t,x,v)dv\right)_\varepsilon$ is tight.

We also have $f_\varepsilon \to f$ in a distributional sense. So we have $\int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\phi(v)dv \to \int_{\mathbb{R}^d} f(t,x,v)\phi(v)dv$, but with partmanteau's theorem it holds also for bounded functions, say the constant function equal to one, so we have $\int_{\mathbb{R}^d} f(t,x,v)dv = \rho(t,x)$. Using Fatou again we get

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)dxdv \leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)dxdv = M^{in}$$

We don't have necessarly equality, since the lack of tightness in the x-variable and the potential loss of mass as $|x| \to \infty$.

We have $\rho_\varepsilon^{in} \to \rho^{in}$ in $L^1(\mathbb{R}^d)$ by the dominated convergence, since $(\rho_\varepsilon^{in})_\varepsilon$ is dominated by ρ^{in}. Using the fact $\zeta_\varepsilon \xrightarrow{D'} \delta_0$ we get

$$E_\varepsilon^{in} = -\zeta_\varepsilon * \nabla G_d * \rho_\varepsilon^{in} \xrightarrow{D'} -\nabla G_d * \rho^{in} = E^{in}$$

We have $E_\varepsilon \to E$ in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ weak* as $\varepsilon \to 0$, this implies

$$\int_a^b \int_{\mathbb{R}^d} E(s,x)\cdot(E_\varepsilon(s,x) - E(s,x))dxds \to 0$$

for all $a<b$ in $\mathbb{R}_+$. Hence,

$$\liminf_{\varepsilon\to 0} \int_a^b\int_{\mathbb{R}^d} |E_\varepsilon(t,x)|^2 dxds \geq \int_a^b\int_{\mathbb{R}^d} |E(t,x)|^2 dxdt$$

by Fatou.

Similarly we have

$$\int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{1}_{|x|+|v|<R} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv\,dt$$

$$\longrightarrow \int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{1}_{|x|+|v|<R} |v|^2 f(t,x,v)\,dx\,dv\,dt$$

$f_\varepsilon \geq 0$ a.e. $\Longrightarrow$

$$\liminf_{\varepsilon\searrow 0} \int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv\,dt$$

$$\geq \int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{1}_{|x|+|v|<R} |v|^2 f(t,x,v)\,dx\,dv\,dt$$

With $R\to\infty$ and Fatou we get

$$\liminf_{\varepsilon\searrow 0} \int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv\,dt$$

$$\geq \int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f(t,x,v)\,dx\,dv\,dt$$

Finally

$$\int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f(t,x,v)\,dx\,dv\,dt + \int_a^b \int_{\mathbb{R}^d} |E(t,x)|^2\,dx\,dt$$

$$\leq \liminf_{\varepsilon\searrow 0} \left(\int_a^b \iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv\,dt + \int_a^b \int_{\mathbb{R}^d} |E_\varepsilon(t,x)|^2\,dx\,dt \right)$$

$$= 2\mathcal{E}^{in}(b-a)$$

for all $a<b$ in $\mathbb{R}_+$. Therefor for almost every $t\in\mathbb{R}_+$.

The last ~~in~~equality used the energy conservation.

We also have

$$E_\varepsilon = -\zeta_\varepsilon * \nabla G_d * \rho_\varepsilon \xrightarrow{D'} -\nabla G_d * \rho ,$$

but also $E_\varepsilon \xrightarrow{D'} E$, so we have $E = -\nabla G_d * \rho$ in $D'(\mathbb{R}^d)$

or $-\nabla_x \Phi = -\nabla G_d * \rho$. With $-\Delta G_d = \delta_0$ we

have $-\Delta_x \Phi = \delta_0 * \rho = \rho = \int_{\mathbb{R}^d} = f\,dv$ in $D'(\mathbb{R}_+ \times \mathbb{R}^d)$

<u>Step 3</u>: Now we look at the Vlasov equation

$$(\partial_t + v\cdot\nabla_x) f_\varepsilon = \operatorname{div}_v (f_\varepsilon \nabla_x \Phi_\varepsilon).$$

The linear part we can pass easily in the sense of distributions

i.e. $$(\partial_t + v\cdot\nabla_x) f_\varepsilon \xrightarrow{D'} (\partial_t + v\cdot\nabla_x) f \quad \text{as } \varepsilon \searrow 0.$$

Now ρ_ε is uniformly bounded in $L^1(\mathbb{R}^d)$ and

$$-\nabla_x^2 \Phi_\varepsilon = \nabla_x^2 G_d * (\zeta_\varepsilon * \zeta_\varepsilon * \rho_\varepsilon),$$

so we can use the a priori estimate on the derivatives of the force field, i.e. for all $t \geq 0$ we have

$$\|\nabla_x^2 \Phi_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq C \|\rho_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq \text{const.} \quad 1 \leq p \leq \frac{d+2}{d}$$

$$\|\partial_t \nabla_x \Phi_\varepsilon(t,\cdot)\|_{L^q(\mathbb{R}^d)} \leq C \|j_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^d)} \leq \text{const.}, \quad 1 < q \leq \frac{d+2}{d+2}$$

Since they are uniform bounds in ε and t we get

$$\sup_{t,\varepsilon>0} \|\partial_t \nabla_x \Phi_\varepsilon(t,\cdot)\|_{L^q(\mathbb{R}^d)} + \|\nabla_x^2 \Phi_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^d)} < \infty$$

for $1 < p \leq \frac{d+2}{d}$ and $1 < q \leq \frac{d+2}{d+1}$

Hence, $-E_\varepsilon = \nabla_x \Phi_\varepsilon \in W^{1,q}$ for $1 < q \leq \frac{d+2}{d+1}$. By Rellich's compactness theorem we have $W^{1,q} \subset\subset L^1$. So we have the strong convergence

$$\chi \nabla_x \Phi_\varepsilon \to \chi \nabla_x \Phi$$

in $L^1(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d)$, since $E_\varepsilon \to E = -\nabla_x \Phi$ in $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ weak-* as $\varepsilon \searrow 0$, for all $\chi \in C_c(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d)$

So

$$\iiint_{\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d} \chi f_\varepsilon(t,x,v) \nabla_x \Phi_\varepsilon(t,x) \, dx \, dv \, dt$$

$$\xrightarrow{\varepsilon \searrow 0} \iiint_{\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d} \chi f(t,x,v) \nabla_x \Phi(t,x) \, dx \, dv \, dt$$

or

$$f_\varepsilon \nabla_x \Phi_\varepsilon \xrightarrow{\mathcal{D}'} f \nabla_x \Phi \quad \text{as } \varepsilon \searrow 0$$

Hence,

$$(\partial_t + v \cdot \nabla_x) f = \operatorname{div}_v (f \nabla_x \Phi).$$

since div_v is a linear operator.

<u>Step 4:</u> First we have

$$\partial_t f_\varepsilon = -\operatorname{div}_x (v f_\varepsilon) + \operatorname{div}_v (f_\varepsilon \nabla_x \Phi_\varepsilon).$$

Next we have

$$\iint_{|x|,|v|\geq 1} |v| f_\varepsilon(t,x,v)\,dx\,dv \leq \iint_{|x|,|v|\geq 1} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv \leq 2\,\mathcal{E}^{in}$$

and

$$\iint_{|x|,|v|\leq 1} |v| f_\varepsilon(t,x,v)\,dx\,dv \leq \iint_{|x|,|v|\leq 1} f_\varepsilon(t,x,v)\,dx\,dv \leq M^{in}$$

These bounds are independent of ε and t so we have

$$A := \sup_{t,\varepsilon>0} \|v f_\varepsilon(t,\cdot,\cdot)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} < \infty$$

On the other hand we have

$$\int_{\mathbb{R}^d} f_\varepsilon(t,x,v)^2 |\nabla_x \Phi_\varepsilon(t,x)|^2 dx$$

$$\leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \int_{\mathbb{R}^d} |\nabla_x \Phi_\varepsilon|^2 dx$$

$$\leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \cdot 2\,\mathcal{E}^{in}$$

where we used $0 \leq f_\varepsilon(t,x,v) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}$ from the previous section. Hence,

$$B := \sup_{t,\varepsilon>0} \|f_\varepsilon(t,\cdot,\cdot)\nabla_x\Phi_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^d_v,\,L^2(\mathbb{R}^d))} < \infty.$$

So

$$A + B < \infty.$$

Now let $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. We have

$$\cancel{\sup} \left| \frac{d}{dt} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t,x,v)\chi(x,v)\,dx\,dv \right|$$

$$= \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} \partial_t f_\varepsilon(t,x,v)\chi(x,v)\,dx\,dv \right|$$

$$= \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(-\operatorname{div}_x(v f_\varepsilon) + \operatorname{div}_v(f_\varepsilon \nabla_x \Phi_\varepsilon)\right)(t,x,v)\chi(x,v)\,dx\,dv \right|$$

$$\leq \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} v f_\varepsilon(t,x,v) \operatorname{div}_x(\chi)(x,v)\,dx\,dv \right|$$

$$+ \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t,x,v)\nabla_x \Phi(t,x) \operatorname{div}_v(\chi)(x,v)\,dx\,dv \right|$$

So

$$\sup_{\varepsilon,t>0} \left| \frac{d}{dt} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t,x,v)\chi(x,v)\,dx\,dv \right|$$

$$\leq C_1 \cdot A + \sup_{\varepsilon,t>0} \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t,x,v)\nabla_x \Phi(t,x) \operatorname{div}_v(\chi)(x,v)\,dx\,dv \right|$$

for some constand $C_1 > 0$ since χ is bounded. We know $L^2 \subseteq L^1$ for some finite measure which we basicly have since $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. So we can bound the second term with $C_2 \cdot B$ for some constant $C_2 > 0$. Hence,

$$\sup_{\varepsilon,t>0} \left| \frac{d}{dt} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t,x,v)\chi(x,v)\,dx\,dv \right| \leq C_1 A + C_2 B < \infty$$

since $C_1, C_2 > 0$ are independent of ε & t.

So we can use the Ascoli-Arzela theorem, i.e.

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(t,x,v)\,\chi(x,v)\,dx\,dv \xrightarrow{\varepsilon \searrow 0} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,\chi(x,v)\,dx\,dv$$

uniformly on $[0,T]$ for all $T>0$. Letting $T\to\infty$ we see that

$$t \longmapsto \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,\chi(x,v)\,dx\,dv$$

is continuous on $\mathbb{R}_+$. For $t=0$ we have

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\varepsilon(0,x,v)\,\chi(x,v)\,dx\,dv \xrightarrow{\varepsilon \searrow 0} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(0,x,v)\,\chi(x,v)\,dx\,dv$$

and since ξ_ε is a even mollifier we can use Lemma 4.2.2

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \xi_\varepsilon * (\mathbb{1}_{\varepsilon|x|<1}\,\mathbb{1}_{\varepsilon|v|<1}\,f^{in})(x,v)\,\chi(x,v)\,dx\,dv$$

$$= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{1}_{\varepsilon|x|<1}\,\mathbb{1}_{\varepsilon|v|<1}\,f^{in}(x,v)\,\tilde{\chi}(x,v)\,dx\,dv$$

$$\xrightarrow{\varepsilon \searrow 0} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\,\chi(x,v)\,dx\,dv$$

~~for ε small enough since $\chi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$~~, where

$\tilde{\chi} = \xi_\varepsilon * \chi \in C_c^\infty$ since the convolution is bijective in C_c^∞.

$$= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \xi_\varepsilon * (\mathbb{1}_{\varepsilon|x|<1} \mathbb{1}_{\varepsilon|v|<1} f^{in})(x,v) \chi(x,v) \, dx \, dv$$

$$\xrightarrow{\varepsilon \searrow 0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^{in}(x,v) \chi(x,v) \, dx \, dv$$

Hence,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} f^{in}(x,v) \chi(x,v) \, dx \, dv = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(0,x,v) \chi(x,v) \, dx \, dv.$$

Since $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ was arbitrary we have

$$f|_{t=0} = f^{in}.$$

QED