Tools for PDEs

Prof. Mikaela Iacobelli, Dr. Ludovic Cesbron and Antoine Gagnebin

March 1, 2021

Contents

These few notes present some definitions and theorems that will be useful during the seminar "*Topics in Non-Collisional Kinetic Theory*". These pages are very informal and give an overview of the concepts that we will use. They can also be used as a brief reminder for those who are already familiar with all these concepts.

For a more rigorous and detailed presentation of the definitions of Sobolev spaces, weak solutions and elliptic regularity, it is possible to follow the lecture *Functional Analysis II*, given this spring semester at ETH by Prof. Alessandro Carlotto. The content of this functional analysis course is helpful for the understanding of the seminar but is not compulsory.

It is therefore not necessary to already know all this material, but it should be understood well enough to be used during the semester when needed. Of course this list is not exhaustive and other known results of PDEs may be useful for the understanding of the seminar, but they will be quoted in due course.

1 Functional analysis

1.1 Lebesgue spaces ($\mathbb{R}$-valued)

Definition 1.1. *For $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^d$, $d \geq 1$, we define the Lebesgue space or L^p-space as*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid \int_\Omega |f(x)|^p \, dx < +\infty \right\}.$$

Proposition 1.2. *For $1 \leq p \leq \infty$, the L^p-space, $\big(L^p(\Omega), \|\cdot\|_{L^p(\Omega)}\big)$, is a Banach space with the norm*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Proposition 1.3. *For $1 \leq p \leq \infty$, the dual space of $L^p(\Omega)$ is $(L^p(\Omega))' = L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$. However $(L^\infty(\Omega))' \supsetneq L^1(\Omega)$ and $(L^1(\Omega))' = L^\infty(\Omega)$ if $|\Omega| < +\infty$.*

Corollary 1.4. *$L^2(\Omega)$ is its own dual. In fact $L^2(\Omega)$ is a Hilbert space with inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx.$$

Proposition 1.5 (*Hölder's inequality*)**.** *Let $p, q \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

We can generalized Hölder's inequality:

- *If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then*

$$\|fg\|_{L^r(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

- *If $\sum_{k=1}^n \frac{1}{p_k} = \frac{1}{r}$ and $f_k \in L^{p_k}(\Omega)$. Then*

$$\left\| \prod_{k=1}^n f_k \right\|_{L^r(\Omega)} \leq \prod_{k=1}^n \|f_k\|_{L^{p_k}(\Omega)}.$$

Proposition 1.6 (*Interpolation*)**.** *Let $1 \leq p \leq r \leq q \leq +\infty$ and $\theta \in (0,1)$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. If $f \in L^p(\Omega) \cap L^q(\Omega)$. Then $f \in L^r(\Omega)$ and*

$$\|f\|_{L^r(\Omega)} \leq \|f\|^\theta_{L^p(\Omega)} \, \|f\|^{1-\theta}_{L^q(\Omega)}.$$

Remark 1.7. *If $|\Omega| < +\infty$ then the $L^p(\Omega)$ spaces are a decreasing sequence for inclusion, namely if $p \leq q$ then $L^q(\Omega) \subseteq L^p(\Omega)$.*

Remark 1.8. ***Heuristic :***

1) *The smaller the p the more f has to decay at ∞ to belong to $L^p(\Omega)$.*
2) *The larger the p the more f has to be locally "bounded" to belong to $L^p(\Omega)$.*

Examples 1.9.

1) *Constant functions only belong to $L^\infty(\Omega)$. The function $f(x) = \frac{1}{x}$ decays fast enough to belong to $L^p((1, +\infty))$ $\forall\, p > 1$ but not enough for $L^1((1, +\infty))$.*
2) *The function $f(x) = x^{-\frac{1}{p}}$ goes to $+\infty$ as x goes to 0. The greater the p the more "localized" around 0 the explosion is, namely $x^{-\frac{1}{p}} \in L^q((-1, 1))$ $\forall\, q < p$.*

1.2 Convolution

Definition 1.10. *Two functions f and g, defined almost everywhere (a.e) and measurable on $\mathbb{R}^d$ are called convolutable if for a.e. $x \in \mathbb{R}^d$ the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^d$. The convolution product of f and g is then*

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\ dy, \qquad for\ a.e.\ x \in \mathbb{R}^d.$$

The definition of convolution can be generalised to measures.

Definition 1.11. *Given two measures with bounded variations μ and η, their convolution $\lambda = \mu * \eta$ is a measure such that,*

$$\int_{\mathbb{R}^d} f(x)\ d\lambda(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+y)\ d\mu(x)\ d\eta(y),$$

for all $f = \mathbb{1}_A$, with A a borel set.

Proposition 1.12.

1) *The convolution product is commutative and associative.*

2) ***Young's convolution inequality.***
For $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ and r such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ with $1 \le p, q, r \le +\infty$, we have

$$\|f * g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$

3) ***Regularity and derivatives.***

   a) *If $f \in L^1_{loc}(\mathbb{R}^d)$ and $\phi \in C^0_c(\mathbb{R}^d)$, then f and ϕ are convolutable and the convolution, $f * \phi$, is continuous.*

   b) *If $f \in L^1_{loc}(\mathbb{R}^d)$ and $\phi \in C^k_c(\mathbb{R}^d)$, then $f * \phi \in C^k(\mathbb{R}^d)$ and $\partial^\alpha(f * \phi) = f * \partial^\alpha \phi$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$.*

1.3 Sobolev spaces

Definition 1.13. *Given $\Omega \subseteq \mathbb{R}^d$ open set, $k \in \mathbb{N}$ and $p \in [1, +\infty]$, the Sobolev space, $W^{k,p}(\Omega)$, is defined as*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid \partial^\alpha f \in L^p(\Omega)\ \forall\ |\alpha| \le k\}$$

and equipped with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

Proposition 1.14. *For any k and p, $\big(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}\big)$ is a Banach space.*

Remark 1.15. *When $p = 2$, we write for any $k \in \mathbb{N}$, $W^{k,2}(\Omega) = H^k(\Omega)$ and $H^k(\Omega)$ is a Hilbert space with inner product*

$$\langle f, g\rangle_{H^k(\Omega)} = \sum_{|\alpha| \le k} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2(\Omega)}.$$

1.4 Strong and weak convergence

Let $(E, \|\cdot\|_E)$ be a Banach space, e.g. $E = L^p$ or $W^{k,p}$.

Definition 1.16. *A sequence $(f_n)_{n\in\mathbb{N}} \subset E$ converges strongly to $f \in E$ if*

$$\|f_n - f\|_E \underset{n\to+\infty}{\longrightarrow} 0.$$

Definition 1.17. *A sequence $(f_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $f \in E$ if for all $\phi \in E'$ (the dual space) we have*

$$\langle f_n, \phi\rangle_E \underset{n\to+\infty}{\longrightarrow} \langle f, \phi\rangle_E.$$

Where $\langle \cdot,\cdot\rangle_E$ is the duality bracket on E. And if $(f_n)_{n\in\mathbb{N}}$ converges weakly to f, we write $f_n \rightharpoonup f$.

Proposition 1.18. *Strong convergence implies weak convergence.*

2 Weak derivatives and weak solutions to PDE's

2.1 Weak solutions to PDE's

Motivation : Consider the following PDE with $c \in \mathbb{R}^*$,

$$(T)\begin{cases} \partial_t f(t,x) + c\,\partial_x f(t,x) = 0, & (t,x)\in\mathbb{R}_+\times\mathbb{R},\\ f_{in}(x) = f(0,x), & x\in\mathbb{R}. \end{cases}$$

Assume that $f_{in} \in C^1(\mathbb{R})$. We can show that there exists a unique solution $f \in C^1(\mathbb{R}_+\times\mathbb{R})$ to this system (T).

We can express the fact that $f \in C^1$ is a solution to (T) in the three following equivalent ways :

S1) f satisfies for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\partial_t f(t,x) + c\,\partial_x f(t,x) = 0,$$

and for all $x\in\mathbb{R}$, $f(0,x) = f_{in}(x)$.

S2) The function $(t,x)\mapsto \partial_t f(t,x) + c\,\partial_x f(t,x)$ satisfies for all $\phi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$,

$$\int_{\mathbb{R}_+\times\mathbb{R}} \Big(\partial_t f(t,x) + c\,\partial_x f(t,x)\Big)\phi(t,x)\; dt\; dx = 0,$$

and for all $x\in\mathbb{R}$, $f(0,x) = f_{in}(x)$.

S3) f satisfies for all $\phi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R})$,

$$\int_{\mathbb{R}_+\times\mathbb{R}} f(t,x)\Big(\partial_t\phi(t,x) + c\,\partial_x\phi(t,x)\Big)\; dt\; dx = \int_{\mathbb{R}} f_{in}(x)\phi(0,x)\; dx.$$

Remark 2.1. *Note that S3) can be obtained from S2) by integration by parts using the fact that ϕ is compactly supported in $\mathbb{R}_+\times\mathbb{R}$.*

The essential remark which motived the definition of weak solution is the following : expression S3) does not involve the derivatives of f. In fact, there is no reason a priori for a function f that satisfies S3) to be $C^1(\mathbb{R}_+ \times \mathbb{R})$. Note however that if it is not C^1 then the three expressions above are not equivalent anymore.

Definition 2.2. *For a given $f_{in} \in L^1_{loc}(\mathbb{R})$, we say that f is a weak solution to the PDE (T) if it satisfies S3).*

Proposition 2.3. *All strong solutions (in the sense of S1)) are weak solutions.*

2.2 Weak derivatives

The definition of weak solutions above leads naturally to the definition of weak derivatives since it implies that a function which is not differentiable may be solution to a PDE.

In order to define rigorously weak derivatives we should study Distribution Theory. (See for example *Sandro Salsa, Partial Differential Equations in Action: From Modelling to Theory.*)

Heuristically, the idea is to identify the derivative of a non-differentiable function with a linear operator on the functional space $C^\infty_c(\mathbb{R})$. Since $C^\infty_c(\mathbb{R})$ is a "small" functional space, for instance included in all $C^k(\mathbb{R})$, $L^p(\mathbb{R})$, $W^{k,p}(\mathbb{R})$, ... , its dual is a very "large" space where we can find a generalisation to the notion of "function".

In other words, for a function f, non-differentiable, we define $\partial_x f$ as the linear map

$$\begin{array}{cccc} \partial_x f & : & C^\infty_c(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & & \phi & \longmapsto & -\int_{\mathbb{R}} f(x)\partial_x\phi(x)\ dx. \end{array}$$

If $f \in L^1_{loc}(\mathbb{R})$ there is a unique $v \in L^1_{loc}(\mathbb{R})$ such that for all $\phi \in C^\infty_c(\mathbb{R})$ we have

$$-\int_{\mathbb{R}} f(x)\partial_x\phi(x)\ dx = \int_{\mathbb{R}} v(x)\phi(x)\ dx,$$

we then identify $\partial_x f$ with $v \in L^1_{loc}(\mathbb{R})$. Naturally, if $f \in C^1(\mathbb{R})$ then $v = \partial_x f \in C^0(\mathbb{R})$.

2.3 Sobolev embedding

Definition 2.4. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces such that $X \subseteq Y$, we say that X is continuously embedded in Y if there exists a constant $C > 0$ such that*

$$\|f\|_Y \leq C\|f\|_X, \qquad \forall\ f \in X.$$

We write $X \hookrightarrow Y$.

Theorem 2.5 (*Morrey's Theorem*)**.** *Let $s \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $s > \frac{d}{2} + k$, then*

$$H^s(\mathbb{R}^d) \hookrightarrow C^k(\mathbb{R}^d).$$

Definition 2.6. *For $0 < \alpha \leq 1$, the Hölder space, $C^{0,\alpha}(\mathbb{R}^d)$, consists of all functions, $f \in C^0(\mathbb{R}^d)$ for which the norm*

$$\|f\|_{C^{0,\alpha}(\mathbb{R}^d)} := \|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite.

Theorem 2.7 (*Sobolev embedding Theorem*). *Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$.*

(1) If $p < d$ then,

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d) \quad \text{with } p^* = \frac{dp}{d-p}.$$

p^* is called the Sobolev conjugate.

(2) If $p > d$ then

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow C^{0,\alpha}(\mathbb{R}^d) \quad \text{with } \alpha = 1 - \frac{d}{p}.$$

Theorem 2.8 (*Generalisation of Sobolev embedding Theorem*). *Let $1 \leq p < q < \infty$, $l, k \in \mathbb{N}$ such that $l < k$ and $0 < \alpha \leq 1$.*

(1) If p, q, k and l satisfies $\frac{1}{p} - \frac{k}{d} = \frac{1}{q} - \frac{l}{d}$, then

$$W^{k,p}(\mathbb{R}^d) \subseteq W^{l,q}(\mathbb{R}^d).$$

(2) If $pk > d$ and $\frac{1}{p} - \frac{k}{d} = -\frac{l+\alpha}{d}$, then

$$W^{k,p}(\mathbb{R}^d) \subset C^{l,\alpha}(\mathbb{R}^d).$$

3 Elliptic regularity : Poisson equation

Consider the Poisson equation on $\Omega \subseteq \mathbb{R}^d$,

$$-\Delta u = f.$$

If $\Omega = \mathbb{R}^d$, then we know explicitly the fundamental solution of the Laplace equation, $\Delta u = 0$, for $d \geq 3$,

$$G_d(x) = C_d \frac{1}{|x|^{d-2}}, \qquad \forall x \neq 0.$$

And so the unique C^2 solution to the Poisson equation is given by

$$u(x) = G_d * f(x) = \int_{\mathbb{R}^d} C_d \frac{1}{|x-y|^{d-2}} f(y)\, dy.$$

The regularity of u then follows from the regularity of G_d or f.

If Ω is bounded then we can try to modify the fundamental solution to construct solutions on Ω but the resulting $\widetilde{G_d}$ will not be explicit in general. Instead we adopt a weak solution approach.

Theorem 3.1. *If Ω is a smooth bounded open subset of $\mathbb{R}^d$ and $f \in L^2(\Omega)$ then there exists a unique weak solution $u \in H^1_0(\Omega)$ to*

$$(P) \begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Where $H^1_0(\Omega) = \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$, that is the closure of $C_c^\infty(\Omega)$ under the H^1-norm. We can see $H^1_0(\Omega)$ as $H^1(\Omega)$ such that the functions vanish on the boundary of Ω, denoted $\partial\Omega$.

Furthermore, being solution to (P) implies regularity for u :

Theorem 3.2. *If $u \in H^1(\Omega)$ is a weak solution to (P) with $f \in L^2(\Omega)$, then for all $\Omega' \subset\subset \Omega$ we have $u \in H^2(\Omega')$ and*

$$\|u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

for some constant $C > 0$.

Remark 3.3. *$\Omega' \subset\subset \Omega$ means that Ω' is compactly embedded. Namely every bounded sequence in Ω' has a converging subsequence in Ω.*

Theorem 3.4. *Let $u \in H^1(\Omega)$ be a weak solution to (P). If $f \in H^k(\Omega)$, then $u \in H^{k+2}(\Omega')$ for any $\Omega' \subset\subset \Omega$ and we have*

$$\|u\|_{H^{k+2}(\Omega')} \leq C \left(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

where C depends on dist$(\Omega', \partial\Omega)$.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^d$ open bounded and smooth. If u is a weak H^1-solution to (P) then*

1) If $f \in C^0(\Omega)$ then $u \in C^{1,\alpha}(\Omega')$ for all $\alpha \in (0,1)$ and $\Omega' \subset\subset \Omega$. Moreover

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C \left(\|f\|_{C^0(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

2) If $f \in C^{0,\alpha}(\Omega)$ for $\alpha \in (0,1)$ then $u \in C^{2,\alpha}(\Omega')$ for all $\Omega' \subset\subset \Omega$. Moreover

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C \left(\|f\|_{C^{0,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

3) If $f \in C^{k,\alpha}(\Omega)$ for $k \in \mathbb{N}$, $\alpha \in (0,1)$ then $u \in C^{k+2,\alpha}(\Omega')$ for all $\Omega' \subset\subset \Omega$. Moreover

$$\|u\|_{C^{k+2,\alpha}(\Omega')} \leq C \left(\|f\|_{C^{k,\alpha}(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

Remark 3.6. *Although these estimates concern the local Hölder regularity of u, they involve the global L^2-norm due to the fact that the starting point of the Thm is "u is a weak solution to (P) in $H^1_0(\Omega)$". Note that in both 2) and 3) we prove enough regularity to show that u is a strong solution to (P).*

Remark 3.7. *For $0 < \alpha \leq 1$ and $k \in \mathbb{N}$, the Hölder space, $C^{k,\alpha}(\Omega)$, consists of all functions, $f \in C^k(\Omega)$ for which the norm*

$$\|f\|_{C^{k,\alpha}(\Omega)} := \sum_{|\beta| \leq k} \|D^\beta f\|_{L^\infty(\mathbb{R}^d)} + \sum_{|\beta| \leq k} \sup_{x \neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}$$

is finite.