Exercise sheet 1

Rough Path Theory

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Problem 1

For $\alpha \in (0,1]$, recall the space of α-Hölder continuous paths $\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$, and the associated seminorm

$$\|X\|_\alpha = \sup_{0 \le s < t \le T} \frac{|X_{s,t}|}{|t-s|^\alpha}.$$

Show that $\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ becomes a Banach space when equipped with the norm

$$X \mapsto |X_0| + \|X\|_\alpha.$$

Problem 2

Suppose that $X \in \mathcal{C}^\alpha$ for some $\alpha > 1$. Show that the path X must be equal to a constant.

Problem 3

Let $\alpha \in (0,1)$ and let $X \colon [0,T] \to \mathbb{R}$ be the path given by $X_t = t^\alpha$. Show that $X \in \mathcal{C}^\alpha$, but $X \notin \mathcal{C}^{0,\alpha}$.

Problem 4

Let $X \colon [0,T] \to \mathbb{R}^d$ be a smooth path, and let

$$\mathbb{X}_{s,t} = \int_s^t (X_r - X_s) \otimes \mathrm{d}X_r$$

for all $(s,t) \in \Delta_{[0,T]}$, with the integral being defined in the Riemann–Stieltjes (or Young) sense. Show that Chen's relation:

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + X_{s,u} \otimes X_{u,t} \tag{1}$$

holds for all $0 \le s \le u \le t \le T$.

Problem 5

Convince yourself that the space $\mathcal{C}^\alpha \times \mathcal{C}_2^{2\alpha}$ equipped with the norm

$$(X, \mathbb{X}) \mapsto |X_0| + |||\mathbf{X}|||_\alpha$$

is a Banach space. Then show that the space of rough paths $\mathscr{C}^\alpha$ (i.e. the elements of $\mathcal{C}^\alpha \times \mathcal{C}_2^{2\alpha}$ which are constrained by Chen's relation (1)) is a complete metric space with metric

$$(\mathbf{X}, \tilde{\mathbf{X}}) \mapsto |X_0 - \tilde{X}_0| + \|\mathbf{X}; \tilde{\mathbf{X}}\|_\alpha.$$

Problem 6

Let $\frac{1}{3} < \alpha < \beta \leq \frac{1}{2}$ and let $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C} \times \mathcal{C}_2$. For each $n \geq 1$, let $\mathbf{X}^n = (X^n, \mathbb{X}^n) \in \mathscr{C}^\beta$ be a β-Hölder rough path, and assume that $\sup_{n\geq 1} |||\mathbf{X}^n|||_\beta < \infty$. Suppose further that $X^n \to X$ and $\mathbb{X}^n \to \mathbb{X}$ uniformly as $n \to \infty$.

Show that $\mathbf{X} \in \mathscr{C}^\beta$, and that $\|\mathbf{X}^n; \mathbf{X}\|_\alpha \to 0$ as $n \to \infty$.

Problem 7

Part (a) Suppose that the pair $(X, \mathbb{X})$ satisfies Chen's relation (1). Let $0 \leq s < t \leq T$, and let $\{s = u_0 < u_1 < \cdots < u_N = t\}$ be a partition of the interval $[s, t]$.

Show that

$$\mathbb{X}_{s,t} = \sum_{i=0}^{N-1} \big(\mathbb{X}_{u_i,u_{i+1}} + X_{s,u_i} \otimes X_{u_i,u_{i+1}}\big).$$

Part (b) Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}_g^{0,\alpha}$ be a geometric rough path. Show that

$$\lim_{\delta\to 0} \sup_{|t-s|<\delta} \frac{|X_{s,t}|}{|t-s|^\alpha} = 0, \qquad \text{and} \qquad \lim_{\delta\to 0} \sup_{|t-s|<\delta} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} = 0.$$

Part (c) Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}_g^{0,\frac{1}{2}}$ be a geometric $\frac{1}{2}$-Hölder rough path. Prove (using the results of parts (a) and (b)) that $\mathbb{X}$ is necessarily equal to the Riemann–Stieltjes integral of X against itself, i.e. that

$$\mathbb{X}_{s,t} = \lim_{|\pi|\to 0} \sum_{i=0}^{N-1} X_{s,u_i} \otimes X_{u_i,u_{i+1}},$$

where $\pi = \{s = u_0 < u_1 < \cdots < u_N = t\}$.