Exercise sheet 1 with solutions

Rough Path Theory

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Problem 1

For $\alpha \in (0,1]$, recall the space of α-Hölder continuous paths $\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$, and the associated seminorm

$$\|X\|_\alpha = \sup_{0 \le s < t \le T} \frac{|X_{s,t}|}{|t-s|^\alpha}.$$

Show that $\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ becomes a Banach space when equipped with the norm

$$X \mapsto |X_0| + \|X\|_\alpha.$$

Solution:

It is easy to see that $\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ is a vector space and that $X \mapsto |X_0| + \|X\|_\alpha$ does indeed define a norm on $\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$. It remains to show that this space is complete. To this end, let $(X^n)_{n\ge1}$ be a Cauchy sequence in $\mathcal{C}^\alpha([0,T];\mathbb{R}^d)$, so that

$$|X_0^n - X_0^m| + \|X^n - X^m\|_\alpha \longrightarrow 0 \qquad \text{as} \quad n,m \longrightarrow \infty.$$

For each $t \in [0,T]$, since

$$\begin{aligned} |X_t^n - X_t^m| &\le |X_0^n - X_0^m| + |X_{0,t}^n - X_{0,t}^m| \\ &\le |X_0^n - X_0^m| + \|X^n - X^m\|_\alpha t^\alpha \longrightarrow 0 \qquad \text{as} \quad n,m \longrightarrow \infty, \end{aligned}$$

we can define

$$X_t := \lim_{n\to\infty} X_t^n.$$

For $0 \le s < t \le T$, we have

$$\frac{|X_{s,t}|}{|t-s|^\alpha} = \lim_{n\to\infty} \frac{|X_{s,t}^n|}{|t-s|^\alpha} \le \sup_{n\ge1} \|X^n\|_\alpha,$$

and hence

$$\|X\|_\alpha = \sup_{0\le s<t\le T} \frac{|X_{s,t}|}{|t-s|^\alpha} \le \sup_{n\ge1} \|X^n\|_\alpha < \infty,$$

which means that the path X is itself α-Hölder continuous, i.e. $X \in \mathcal{C}^\alpha$.

We still need to show that $X^n \to X$ in $\mathcal{C}^\alpha$. To see this, note that

$$\frac{|X^n_{s,t} - X_{s,t}|}{|t-s|^\alpha} = \lim_{m\to\infty} \frac{|X^n_{s,t} - X^m_{s,t}|}{|t-s|^\alpha} \le \limsup_{m\to\infty} \|X^n - X^m\|_\alpha,$$

and hence that

$$\|X^n - X\|_\alpha = \sup_{0\le s<t\le T} \frac{|X^n_{s,t} - X_{s,t}|}{|t-s|^\alpha} \le \limsup_{m\to\infty} \|X^n - X^m\|_\alpha \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty.$$

Thus, $X^n \to X$ in $\mathcal{C}^\alpha$ as desired.

Problem 2

Suppose that $X \in \mathcal{C}^\alpha$ for some $\alpha > 1$. Show that the path X must be equal to a constant.

Solution:

Let $t \in [0,T]$ and let $\pi = \{0 = s_0 < s_1 < \cdots < s_N = t\}$ be a partition of the interval $[0,t]$. Then

$$\begin{aligned} |X_{0,t}| &= \left| \sum_{i=0}^{N-1} X_{s_i,s_{i+1}} \right| \le \sum_{i=0}^{N-1} |X_{s_i,s_{i+1}}| \le \|X\|_\alpha \sum_{i=0}^{N-1} |s_{i+1} - s_i|^\alpha \\ &\le \|X\|_\alpha \left(\sum_{i=0}^{N-1} |s_{i+1} - s_i| \right) \max_{0\le i<N} |s_{i+1} - s_i|^{\alpha-1} = \|X\|_\alpha t |\pi|^{\alpha-1}. \end{aligned}$$

Since $\alpha - 1 > 0$ and we can choose the mesh size $|\pi|$ to be arbitrarily small, it follows that $|X_{0,t}| = 0$, and hence that $X_t = X_0$ for all $t \in [0,T]$.

Problem 3

Let $\alpha \in (0,1)$ and let $X \colon [0,T] \to \mathbb{R}$ be the path given by $X_t = t^\alpha$. Show that $X \in \mathcal{C}^\alpha$, but $X \notin \mathcal{C}^{0,\alpha}$.

Solution:

Let $0 \le s < t \le T$. Note that $0 \le \frac{s}{t} < 1$ and $0 < 1 - \frac{s}{t} \le 1$. Then

$$\frac{|X_{s,t}|}{|t-s|^\alpha} = \frac{t^\alpha - s^\alpha}{(t-s)^\alpha} = \frac{1 - (\frac{s}{t})^\alpha}{(1-\frac{s}{t})^\alpha} \le \frac{1 - \frac{s}{t}}{1 - \frac{s}{t}} = 1,$$

so $\|X\|_\alpha \le 1$, and hence $X \in \mathcal{C}^\alpha$.

Note that

$$\frac{|X_{0,t}|}{|t-0|^\alpha} = \frac{t^\alpha}{t^\alpha} = 1$$

for any $t \in (0,T]$, which means that

$$\lim_{\delta\to 0} \sup_{|t-s|<\delta} \frac{|X_{s,t}|}{|t-s|^\alpha} \ge 1.$$

Thus, by the characterization of $\mathcal{C}^{0,\alpha}$ given in the lectures, it follows that $X \notin \mathcal{C}^{0,\alpha}$.

Problem 4

Let $X\colon [0,T] \to \mathbb{R}^d$ be a smooth path, and let

$$\mathbb{X}_{s,t} = \int_s^t (X_r - X_s) \otimes \mathrm{d}X_r$$

for all $(s,t) \in \Delta_{[0,T]}$, with the integral being defined in the Riemann–Stieltjes (or Young) sense. Show that Chen's relation:

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + X_{s,u} \otimes X_{u,t} \tag{1}$$

holds for all $0 \leq s \leq u \leq t \leq T$.

Solution: Easy

Problem 5

Convince yourself that the space $\mathcal{C}^\alpha \times \mathcal{C}_2^{2\alpha}$ equipped with the norm

$$(X, \mathbb{X}) \mapsto |X_0| + |||\mathbf{X}|||_\alpha$$

is a Banach space. Then show that the space of rough paths $\mathscr{C}^\alpha$ (i.e. the elements of $\mathcal{C}^\alpha \times \mathcal{C}_2^{2\alpha}$ which are constrained by Chen's relation (1)) is a complete metric space with metric

$$(\mathbf{X}, \tilde{\mathbf{X}}) \mapsto |X_0 - \tilde{X}_0| + \|\mathbf{X}; \tilde{\mathbf{X}}\|_\alpha.$$

Solution:

That $\mathcal{C}^\alpha \times \mathcal{C}_2^{2\alpha}$ is a Banach space follows from the same argument as in Problem 1. It remains to show that the space $\mathscr{C}^\alpha$ of rough paths is a closed subset of $\mathcal{C}^\alpha \times \mathcal{C}_2^{2\alpha}$.

To this end, take a sequence $\mathbf{X}^n = (X^n, \mathbb{X}^n)$, $n \geq 1$, of elements satisfying Chen's relation, and such that $|X_0^n - X_0| + \|\mathbf{X}^n; \mathbf{X}\|_\alpha \to 0$ as $n \to \infty$, for some element $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha \times \mathcal{C}_2^{2\alpha}$. We have that

$$\mathbb{X}^n_{s,t} = \mathbb{X}^n_{s,u} + \mathbb{X}^n_{u,t} + X^n_{s,u} \otimes X^n_{u,t}$$

for all triplets $s \leq u \leq t$ and all $n \geq 1$. Letting $n \to \infty$, each term converges, and we obtain

$$\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + X_{s,u} \otimes X_{u,t}.$$

That is, Chen's relation holds for $\mathbf{X}$.

Problem 6

Let $\frac{1}{3} < \alpha < \beta \leq \frac{1}{2}$ and let $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C} \times \mathcal{C}_2$. For each $n \geq 1$, let $\mathbf{X}^n = (X^n, \mathbb{X}^n) \in \mathscr{C}^\beta$ be a β-Hölder rough path, and assume that $\sup_{n \geq 1} |||\mathbf{X}^n|||_\beta < \infty$. Suppose further that $X^n \to X$ and $\mathbb{X}^n \to \mathbb{X}$ uniformly as $n \to \infty$.

Show that $\mathbf{X} \in \mathscr{C}^\beta$, and that $\|\mathbf{X}^n; \mathbf{X}\|_\alpha \to 0$ as $n \to \infty$.

Solution:

The results on lower semi-continuity and interpolation of Hölder continuous paths are easily seen to also be valid for two-parameter functions $\mathbb{X}\colon \Delta_{[0,T]} \to \mathbb{R}^{d \times d}$. It is then easy to deduce that $\|\mathbf{X}\|_\beta < \infty$ and that $\|\mathbf{X}^n; \mathbf{X}\|_\alpha \to 0$ as $n \to \infty$. It follows from the previous problem that Chen's relation holds for $(X, \mathbb{X})$, and hence that $\mathbf{X} \in \mathscr{C}^\beta$.

Problem 7

Part (a) Suppose that the pair $(X, \mathbb{X})$ satisfies Chen's relation (1). Let $0 \le s < t \le T$, and let $\{s = u_0 < u_1 < \cdots < u_N = t\}$ be a partition of the interval $[s, t]$.

Show that

$$\mathbb{X}_{s,t} = \sum_{i=0}^{N-1} \big(\mathbb{X}_{u_i,u_{i+1}} + X_{s,u_i} \otimes X_{u_i,u_{i+1}}\big).$$

Part (b) Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}_g^{0,\alpha}$ be a geometric rough path. Show that

$$\lim_{\delta\to 0} \sup_{|t-s|<\delta} \frac{|X_{s,t}|}{|t-s|^\alpha} = 0, \qquad \text{and} \qquad \lim_{\delta\to 0} \sup_{|t-s|<\delta} \frac{|\mathbb{X}_{s,t}|}{|t-s|^{2\alpha}} = 0.$$

Part (c) Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}_g^{0,\frac{1}{2}}$ be a geometric $\frac{1}{2}$-Hölder rough path. Prove (using the results of parts (a) and (b)) that $\mathbb{X}$ is necessarily equal to the Riemann–Stieltjes integral of X against itself, i.e. that

$$\mathbb{X}_{s,t} = \lim_{|\pi|\to 0} \sum_{i=0}^{N-1} X_{s,u_i} \otimes X_{u_i,u_{i+1}},$$

where $\pi = \{s = u_0 < u_1 < \cdots < u_N = t\}$.

Solution:

Part (a) One can show by induction on $k = 0, 1, \ldots, N-1$ that

$$\mathbb{X}_{s,u_{k+1}} = \sum_{i=0}^{k} \big(\mathbb{X}_{u_i,u_{i+1}} + X_{s,u_i} \otimes X_{u_i,u_{i+1}}\big).$$

Setting $k = N-1$ then gives the desired equality.

Part (b) Since $X \in \mathcal{C}^{0,\alpha}$, the first limit was proved in the lecture notes. The second limit follows from a nearly identical argument applied to $\mathbb{X}$. In particular, for any $\varepsilon > 0$ we know there exists a smooth path Y such that $\|\mathbb{X} - \mathbb{Y}\|_{2\alpha} \le \|\mathbf{X}; \mathbf{Y}\|_\alpha < \varepsilon$, where $\mathbf{Y} = (Y, \mathbb{Y})$ and $\mathbb{Y}_{s,t} = \int_s^t (Y_r - Y_s) \otimes \mathrm{d}Y_r$. We can bound $|\mathbb{Y}_{s,t}|/|t-s|^{2\alpha}$ by an identical argument to the one in the lecture notes.

Part (c) Let $\varepsilon > 0$. By part (b) (with $\alpha = \frac{1}{2}$), there exists a $\delta > 0$ such that

$$\sup_{|v-u|<\delta} \frac{|\mathbb{X}_{u,v}|}{|v-u|} < \varepsilon.$$

Let $\pi = \{s = u_0 < u_1 < \cdots < u_N = t\}$ be a partition of the interval $[s, t]$ with mesh size $|\pi| < \delta$. Then, using part (a), we have

$$\left|\mathbb{X}_{s,t} - \sum_{i=0}^{N-1} X_{s,u_i} \otimes X_{u_i,u_{i+1}}\right| = \left|\sum_{i=0}^{N-1} \mathbb{X}_{u_i,u_{i+1}}\right| < \sum_{i=0}^{N-1} \varepsilon |u_{i+1} - u_i| = \varepsilon |t-s|.$$

Since $\varepsilon > 0$ may be taken arbitrarily small, the result follows.