Exercise sheet 2

Rough Path Theory

ETH Zürich

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Problem 1

Let $\beta \in (\frac{1}{3}, \frac{1}{2}]$, and let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\beta$ be a (for simplicity one-dimensional) β-Hölder rough path. Let B be a one-dimensional standard Brownian motion, and let

$$\mathbb{B}_{s,t} = \int_s^t B_{s,r} \, \mathrm{d}B_r$$

be the Itô rough path lift of B.

Note that, since X is a continuous deterministic path, $\int_s^t X_{s,r} \, \mathrm{d}B_r$ is a well-defined Itô integral. It is not clear that one can directly define the integral of B against X. However, by imposing integration by parts, we can define

$$\int_s^t B_{s,r} \, \mathrm{d}X_r := X_{s,t} B_{s,t} - \int_s^t X_{s,r} \, \mathrm{d}B_r.$$

Let

$$Z_t = \begin{pmatrix} X_t \\ B_t \end{pmatrix}, \qquad \mathbb{Z}_{s,t} = \begin{pmatrix} \mathbb{X}_{s,t} & \int_s^t X_{s,r} \, \mathrm{d}B_r \\ \int_s^t B_{s,r} \, \mathrm{d}X_r & \mathbb{B}_{s,t} \end{pmatrix}.$$

Part (a) Show that $\mathbf{Z} = (Z, \mathbb{Z})$ satisfies Chen's relation almost surely.

Part (b) Let $q > 2$. Show that

$$\left\| \int_s^t X_{s,r} \, \mathrm{d}B_r \right\|_{L^{q/2}} \leq C|t-s|^{\frac{1}{2}+\beta}, \qquad \left\| \int_s^t B_{s,r} \, \mathrm{d}X_r \right\|_{L^{q/2}} \leq C|t-s|^{\frac{1}{2}+\beta}$$

for some constant C.

Part (c) Use the Kolmogorov criterion for rough paths to show that $\mathbf{Z}$ is an α-Hölder rough path for any $\alpha \in (\frac{1}{3}, \beta)$.

Problem 2

Let $\alpha, \gamma \in (0,1]$ such that $\alpha(1+\gamma) > 1$. Let $X \in \mathcal{C}^\alpha([0,T];\mathbb{R}^d)$ and $f \in C^{1+\gamma}(\mathbb{R}^d;\mathbb{R})$. Prove that $\int_0^T Df(X_u) \, \mathrm{d}X_u$ is a well-defined Young integral, and that

$$f(X_T) = f(X_0) + \int_0^T Df(X_u) \, \mathrm{d}X_u.$$

Problem 3

Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and $X \in \mathcal{C}^\alpha$. Convince yourself that the space $\mathscr{D}_X^{2\alpha}$ of controlled paths with respect to X, when equipped with the norm

$$\|Y, Y'\|_{\mathscr{D}_X^{2\alpha}} = |Y_0| + |Y_0'| + \|Y'\|_\alpha + \|R^Y\|_{2\alpha},$$

becomes a Banach space.

Problem 4

For some $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, let $F \in \mathcal{C}^{2\alpha}$ be a 2α-Hölder continuous path, and let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ and $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$ be two rough paths such that

$$\tilde{X}_t = X_t, \qquad \tilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + F_{s,t} \qquad \text{for all} \quad s \leq t.$$

Let $(Y, Y') \in \mathscr{D}_X^{2\alpha} = \mathscr{D}_{\tilde{X}}^{2\alpha}$. Show that

$$\int_0^T Y_u \, \mathrm{d}\tilde{\mathbf{X}}_u = \int_0^T Y_u \, \mathrm{d}\mathbf{X}_u + \int_0^T Y_u' \, \mathrm{d}F_u.$$

Problem 5

Let $\frac{1}{3} < \alpha \leq \frac{1}{2}$ and $0 < \beta \leq \alpha$ such that $2\alpha + \beta > 1$, and define $\gamma = \alpha + \beta$. Let $X \in \mathcal{C}^\alpha$. Let's say that a pair (Y, Y') is a (β, γ)-controlled path if $Y \in \mathcal{C}^\alpha$, $Y' \in \mathcal{C}^\beta$ and $R^Y \in \mathcal{C}_2^\gamma$, where R^Y is defined by

$$Y_{s,t} = Y_s' X_{s,t} + R_{s,t}^Y.$$

Part (a) Let $f \in C^{1+\beta/\alpha}$. Show that $(f(X), Df(X))$ is a (β, γ)-controlled path.

Part (b) Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ be a rough path, and let (Y, Y') be a (β, γ)-controlled path. Use the sewing lemma to prove that the limit

$$\int_0^t Y_u \, \mathrm{d}\mathbf{X}_u := \lim_{|\pi| \to 0} \sum_{[u,v] \in \pi} Y_u X_{u,v} + Y_u' \mathbb{X}_{u,v}$$

exists.