Exercise sheet 2 with solutions

Rough Path Theory

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Problem 1

Let $\beta \in (\frac{1}{3}, \frac{1}{2}]$, and let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\beta$ be a (for simplicity one-dimensional) β-Hölder rough path. Let B be a one-dimensional standard Brownian motion, and let

$$\mathbb{B}_{s,t} = \int_s^t B_{s,r} \, \mathrm{d}B_r$$

be the Itô rough path lift of B.

Note that, since X is a continuous deterministic path, $\int_s^t X_{s,r} \, \mathrm{d}B_r$ is a well-defined Itô integral. It is not clear that one can directly define the integral of B against X. However, by imposing integration by parts, we can define

$$\int_s^t B_{s,r} \, \mathrm{d}X_r := X_{s,t} B_{s,t} - \int_s^t X_{s,r} \, \mathrm{d}B_r.$$

Let

$$Z_t = \begin{pmatrix} X_t \\ B_t \end{pmatrix}, \qquad \mathbb{Z}_{s,t} = \begin{pmatrix} \mathbb{X}_{s,t} & \int_s^t X_{s,r} \, \mathrm{d}B_r \\ \int_s^t B_{s,r} \, \mathrm{d}X_r & \mathbb{B}_{s,t} \end{pmatrix}.$$

Part (a) Show that $\mathbf{Z} = (Z, \mathbb{Z})$ satisfies Chen's relation almost surely.

Part (b) Let $q > 2$. Show that

$$\left\| \int_s^t X_{s,r} \, \mathrm{d}B_r \right\|_{L^{q/2}} \leq C|t-s|^{\frac{1}{2}+\beta}, \qquad \left\| \int_s^t B_{s,r} \, \mathrm{d}X_r \right\|_{L^{q/2}} \leq C|t-s|^{\frac{1}{2}+\beta}$$

for some constant C.

Part (c) Use the Kolmogorov criterion for rough paths to show that $\mathbf{Z}$ is an α-Hölder rough path for any $\alpha \in (\frac{1}{3}, \beta)$.

Solution:

Part (a) We already know that $(Z^1, \mathbb{Z}^{11}) = (X, \mathbb{X})$ and $(Z^2, \mathbb{Z}^{22}) = (B, \mathbb{B})$ satisfy Chen's relation. It remains to check the cross terms. This is not difficult. The calculation for $\mathbb{Z}^{21}$ is made easier by first noticing that

$$\begin{aligned} \int_s^t B_{s,r} \, \mathrm{d}X_r &= X_{s,t} B_{s,t} - \int_s^t X_{s,r} \, \mathrm{d}B_r \\ &= \int_s^t X_{s,t} \, \mathrm{d}B_r - \int_s^t X_{s,r} \, \mathrm{d}B_r = \int_s^t X_{r,t} \, \mathrm{d}B_r. \end{aligned} \tag{1}$$

Part (b) By the Burkholder–Davis–Gundy inequality, we have

$$\mathbb{E}\Big[\Big|\int_s^t X_{s,r}\,\mathrm{d}B_r\Big|^{\frac{q}{2}}\Big] \lesssim \Big(\int_s^t (X_{s,r})^2\,\mathrm{d}r\Big)^{\frac{q}{4}} \lesssim \Big(\int_s^t |r-s|^{2\beta}\,\mathrm{d}r\Big)^{\frac{q}{4}} \lesssim |t-s|^{(1+2\beta)\frac{q}{4}},$$

and hence that

$$\Big\|\int_s^t X_{s,r}\,\mathrm{d}B_r\Big\|_{L^{q/2}} \lesssim |t-s|^{\frac{1}{2}+\beta}.$$

Using the expression on the right-hand side of (1), the second estimate follows by a nearly identical argument.

Part (c) It is clear that $\|X_{s,t}\|_{L^q} \lesssim |t-s|^{\beta}$ and $\|\mathbb{X}_{s,t}\|_{L^{q/2}} \lesssim |t-s|^{2\beta}$, and it was shown in the lectures that $\|B_{s,t}\|_{L^q} \lesssim |t-s|^{\frac{1}{2}}$ and $\|\mathbb{B}_{s,t}\|_{L^{q/2}} \lesssim |t-s|$. Combining these estimates with the ones in the previous part, we have that

$$\|Z_{s,t}\|_{L^q} \lesssim |t-s|^{\beta}, \qquad \|\mathbb{Z}_{s,t}\|_{L^{q/2}} \lesssim |t-s|^{2\beta}.$$

It then follows from the Kolmogorov criterion for rough paths that $\mathbf{Z}$ is an α-Hölder rough path for any $\alpha \in (\frac{1}{3}, \beta - \frac{1}{q})$. It remains to let $q \to \infty$.

Problem 2

Let $\alpha, \gamma \in (0,1]$ such that $\alpha(1+\gamma) > 1$. Let $X \in \mathcal{C}^{\alpha}([0,T];\mathbb{R}^d)$ and $f \in C^{1+\gamma}(\mathbb{R}^d;\mathbb{R})$. Prove that $\int_0^T Df(X_u)\,\mathrm{d}X_u$ is a well-defined Young integral, and that

$$f(X_T) = f(X_0) + \int_0^T Df(X_u)\,\mathrm{d}X_u.$$

Solution:

We first note that, since $Df \in C^{\gamma}$, we have

$$|Df(X_t) - Df(X_s)| \lesssim |X_{s,t}|^{\gamma} \lesssim |t-s|^{\alpha\gamma},$$

so that $Df(X) \in \mathcal{C}^{\alpha\gamma}$. Since $\alpha + \alpha\gamma > 1$, we know that the Young integral

$$\int_0^T Df(X_u)\,\mathrm{d}X_u = \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} Df(X_s)X_{s,t}$$

exists. We have

$$f(X_t) - f(X_s) - Df(X_s)X_{s,t} = \int_0^1 \big(Df(X_s + rX_{s,t}) - Df(X_s)\big)X_{s,t}\,\mathrm{d}r,$$

and hence

$$\big|f(X_t) - f(X_s) - Df(X_s)X_{s,t}\big| \lesssim |X_{s,t}|^{1+\gamma}.$$

Let π be a partition of $[0,T]$. Then

$$\begin{aligned}\Big|f(X_T) - f(X_0) - \int_0^T Df(X_u)\,\mathrm{d}X_u\Big| &= \lim_{|\pi|\to 0}\Big|\sum_{[s,t]\in\pi}\big(f(X_t) - f(X_s) - Df(X_s)X_{s,t}\big)\Big| \\ &\lesssim \lim_{|\pi|\to 0}\sum_{[s,t]\in\pi}|X_{s,t}|^{1+\gamma} \\ &\lesssim \lim_{|\pi|\to 0}\sum_{[s,t]\in\pi}|t-s|^{\alpha(1+\gamma)} = 0.\end{aligned}$$

Problem 3

Let $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ and $X \in \mathcal{C}^\alpha$. Convince yourself that the space $\mathscr{D}_X^{2\alpha}$ of controlled paths with respect to X, when equipped with the norm

$$\|Y, Y'\|_{\mathscr{D}_X^{2\alpha}} = |Y_0| + |Y_0'| + \|Y'\|_\alpha + \|R^Y\|_{2\alpha},$$

becomes a Banach space.

Solution: Easy.

Problem 4

For some $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, let $F \in \mathcal{C}^{2\alpha}$ be a 2α-Hölder continuous path, and let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ and $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\mathbb{X}}) \in \mathscr{C}^\alpha$ be two rough paths such that

$$\tilde{X}_t = X_t, \qquad \tilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + F_{s,t} \qquad \text{for all} \quad s \le t.$$

Let $(Y, Y') \in \mathscr{D}_X^{2\alpha} = \mathscr{D}_{\tilde{X}}^{2\alpha}$. Show that

$$\int_0^T Y_u \,\mathrm{d}\tilde{\mathbf{X}}_u = \int_0^T Y_u \,\mathrm{d}\mathbf{X}_u + \int_0^T Y_u' \,\mathrm{d}F_u.$$

Solution:

Since $Y' \in \mathcal{C}^\alpha$ and $F \in \mathcal{C}^{2\alpha}$, and $\alpha + 2\alpha > 1$, we know that

$$\int_0^T Y_u' \,\mathrm{d}F_u = \lim_{|\pi| \to 0} \sum_{[s,t] \in \pi} Y_s' F_{s,t}$$

exists as a Young integral. We have

$$\begin{aligned}\int_0^T Y_u \,\mathrm{d}\tilde{\mathbf{X}}_u &= \lim_{|\pi| \to 0} \sum_{[s,t] \in \pi} Y_s \tilde{X}_{s,t} + Y_s' \tilde{\mathbb{X}}_{s,t} \\ &= \lim_{|\pi| \to 0} \sum_{[s,t] \in \pi} Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t} + Y_s' F_{s,t} \\ &= \int_0^T Y_u \,\mathrm{d}\mathbf{X}_u + \int_0^T Y_u' \,\mathrm{d}F_u.\end{aligned}$$

Problem 5

Let $\frac{1}{3} < \alpha \le \frac{1}{2}$ and $0 < \beta \le \alpha$ such that $2\alpha + \beta > 1$, and define $\gamma = \alpha + \beta$. Let $X \in \mathcal{C}^\alpha$. Let's say that a pair (Y, Y') is a (β, γ)-controlled path if $Y \in \mathcal{C}^\alpha$, $Y' \in \mathcal{C}^\beta$ and $R^Y \in \mathcal{C}_2^\gamma$, where R^Y is defined by

$$Y_{s,t} = Y_s' X_{s,t} + R_{s,t}^Y.$$

Part (a) Let $f \in C^{1+\beta/\alpha}$. Show that $(f(X), Df(X))$ is a (β, γ)-controlled path.

Part (b) Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ be a rough path, and let (Y, Y') be a (β, γ)-controlled path. Use the sewing lemma to prove that the limit

$$\int_0^t Y_u \,\mathrm{d}\mathbf{X}_u := \lim_{|\pi| \to 0} \sum_{[u,v] \in \pi} Y_u X_{u,v} + Y_u' \mathbb{X}_{u,v}$$

exists.

Solution:

Part (a) It is clear that $f(X) \in \mathcal{C}^\alpha$. We have $|Df(X_t) - Df(X_s)| \lesssim |X_{s,t}|^{\beta/\alpha} \lesssim |t-s|^\beta$, so that $Df(X) \in \mathcal{C}^\beta$. Let

$$R^{f(X)}_{s,t} := f(X_t) - f(X_s) - Df(X_s)X_{s,t}.$$

Then

$$\begin{aligned} |R^{f(X)}_{s,t}| &= \big|f(X_t) - f(X_s) - Df(X_s)X_{s,t}\big| \\ &= \left| \int_0^1 \big(Df(X_s + rX_{s,t}) - Df(X_s)\big) X_{s,t} \, \mathrm{d}r \right| \\ &\lesssim |X_{s,t}|^{1+\beta/\alpha} \lesssim |t-s|^\gamma, \end{aligned}$$

so that $R^{f(X)} \in \mathcal{C}^\gamma_2$.

Part (b) Let $A_{s,t} = Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}$, and let $\delta A_{s,u,t} = A_{s,t} - A_{s,u} - A_{u,t}$ for $s \le u \le t$. Exactly as in the lectures, one can show using Chen's relation that

$$\delta A_{s,u,t} = -R^Y_{s,u} X_{u,t} - Y'_{s,u} \mathbb{X}_{u,t},$$

and hence

$$|\delta A_{s,u,t}| = |R^Y_{s,u} X_{u,t} + Y'_{s,u} \mathbb{X}_{u,t}| \lesssim |t-s|^{\gamma+\alpha} + |t-s|^{\beta+2\alpha}.$$

Since $\gamma + \alpha = \beta + 2\alpha > 1$, it follows from the sewing lemma that the desired limit does indeed exist.