# Exercise sheet 3

# Rough Path Theory

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**Problem 1**

Recall that, for a rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, the bracket of $\mathbf{X}$ is defined as the path $[\mathbf{X}] \colon [0,T] \to \mathbb{R}^{d\times d}$ given by

$$[\mathbf{X}]_t := X_{0,t} \otimes X_{0,t} - 2\,\mathrm{Sym}(\mathbb{X}_{0,t}).$$

Show that

$$[\mathbf{X}]_{s,t} = X_{s,t} \otimes X_{s,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t})$$

for all $(s,t) \in \Delta_{[0,T]}$.

**Problem 2**

Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ be a rough path, and let $\Gamma \in \mathcal{C}^{2\alpha}$. Let $Z_t = X_t + \Gamma_t$ for $t \in [0,T]$, and

$$\mathbb{Z}_{s,t} = \mathbb{X}_{s,t} + \int_s^t X_{s,u} \otimes \mathrm{d}\Gamma_u + \int_s^t \Gamma_{s,u} \otimes \mathrm{d}X_u + \int_s^t \Gamma_{s,u} \otimes \mathrm{d}\Gamma_u$$

for $(s,t) \in \Delta_{[0,T]}$, where the three integrals on the right-hand side are defined as Young integrals.

**Part (a)** Show that $\mathbf{Z} = (Z, \mathbb{Z})$ is a rough path.

**Part (b)** Show that $[\mathbf{Z}] = [\mathbf{X}]$.

**Part (c)** Deduce that $\mathbf{Z}$ is weakly geometric if and only if $\mathbf{X}$ is weakly geometric.

**Problem 3**

Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ be a rough path, and suppose that $(Y, Y') \in \mathscr{D}_X^{2\alpha}$ and $(Y', Y'') \in \mathscr{D}_X^{2\alpha}$ are controlled paths. Suppose further that

$$Y_t = Y_0 + \int_0^t Y'_s \,\mathrm{d}\mathbf{X}_s + \Gamma_t$$

for all $t \in [0,T]$, for some path $\Gamma \in \mathcal{C}^{2\alpha}$. Let $f \in C^3$.

Prove that

$$f(Y_T) = f(Y_0) + \int_0^T Df(Y_u)Y'_u \,\mathrm{d}\mathbf{X}_u + \int_0^T Df(Y_u)\,\mathrm{d}\Gamma_u + \frac{1}{2}\int_0^T D^2 f(Y_u)(Y'_u \otimes Y'_u)\,\mathrm{d}[\mathbf{X}]_u.$$

**Problem 4**

Suppose that $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ and $(K, K') \in \mathscr{D}_X^{2\alpha}$ are such that the rough integral $\int_0^\cdot K_u \, \mathrm{d}\mathbf{X}_u$ takes values in $\mathbb{R}$. Let $V$ be the path given by

$$V_t = \exp\left( \int_0^t K_u \, \mathrm{d}\mathbf{X}_u - \frac{1}{2} \int_0^t (K_u \otimes K_u) \, \mathrm{d}[\mathbf{X}]_u \right), \qquad t \in [0, T].$$

Prove that $V$ is the unique solution of the rough differential equation

$$V_t = 1 + \int_0^t V_u K_u \, \mathrm{d}\mathbf{X}_u, \qquad t \in [0, T].$$