Exercise sheet 3 with solutions

Rough Path Theory

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Problem 1

Recall that, for a rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$, the bracket of $\mathbf{X}$ is defined as the path $[\mathbf{X}]\colon [0,T] \to \mathbb{R}^{d\times d}$ given by

$$[\mathbf{X}]_t := X_{0,t} \otimes X_{0,t} - 2\,\mathrm{Sym}(\mathbb{X}_{0,t}).$$

Show that

$$[\mathbf{X}]_{s,t} = X_{s,t} \otimes X_{s,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t})$$

for all $(s,t) \in \Delta_{[0,T]}$.

Solution:

We have

$$\begin{aligned}
[\mathbf{X}]_{s,t} &= [\mathbf{X}]_t - [\mathbf{X}]_s \\
&= X_{0,t} \otimes X_{0,t} - X_{0,s} \otimes X_{0,s} - 2\,\mathrm{Sym}(\mathbb{X}_{0,t} - \mathbb{X}_{0,s}) \\
&= X_{0,t} \otimes X_{0,t} - X_{0,s} \otimes X_{0,s} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t} + X_{0,s} \otimes X_{s,t}) \\
&= X_{0,t} \otimes X_{0,t} - X_{0,s} \otimes X_{0,s} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t}) - X_{0,s} \otimes X_{s,t} - X_{s,t} \otimes X_{0,s} \\
&= X_{0,t} \otimes X_{0,t} - X_{0,s} \otimes X_{0,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t}) - X_{s,t} \otimes X_{0,s} \\
&= X_{s,t} \otimes X_{0,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t}) - X_{s,t} \otimes X_{0,s} \\
&= X_{s,t} \otimes X_{s,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t}).
\end{aligned}$$

Problem 2

Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ be a rough path, and let $\Gamma \in \mathcal{C}^{2\alpha}$. Let $Z_t = X_t + \Gamma_t$ for $t \in [0,T]$, and

$$\mathbb{Z}_{s,t} = \mathbb{X}_{s,t} + \int_s^t X_{s,u} \otimes \mathrm{d}\Gamma_u + \int_s^t \Gamma_{s,u} \otimes \mathrm{d}X_u + \int_s^t \Gamma_{s,u} \otimes \mathrm{d}\Gamma_u$$

for $(s,t) \in \Delta_{[0,T]}$, where the three integrals on the right-hand side are defined as Young integrals.

Part (a) Show that $\mathbf{Z} = (Z, \mathbb{Z})$ is a rough path.

Part (b) Show that $[\mathbf{Z}] = [\mathbf{X}]$.

Part (c) Deduce that $\mathbf{Z}$ is weakly geometric if and only if $\mathbf{X}$ is weakly geometric.

Solution:

Part (a) From the estimate for Young integrals which we got directly from the sewing lemma, we have that

$$\left|\int_s^t X_{s,u} \otimes \mathrm{d}\Gamma_u\right| = \left|\int_s^t X_u \otimes \mathrm{d}\Gamma_u - X_s \otimes \Gamma_{s,t}\right| \lesssim |t-s|^{3\alpha},$$

and similarly

$$\left|\int_s^t \Gamma_{s,u} \otimes \mathrm{d}X_u\right| \lesssim |t-s|^{3\alpha}, \qquad \left|\int_s^t \Gamma_{s,u} \otimes \mathrm{d}\Gamma_u\right| \lesssim |t-s|^{4\alpha}.$$

It is then clear that $|\mathbb{Z}_{s,t}| \lesssim |t-s|^{2\alpha}$, so that $\mathbb{Z} \in \mathcal{C}_2^{2\alpha}$.

It remains to check that $(Z, \mathbb{Z})$ satisfies Chen's relation, which is an easy calculation.

Part (b) By the integration by parts formula for Young integration, we deduce that

$$\mathrm{Sym}\left(\int_s^t X_{s,u} \otimes \mathrm{d}\Gamma_u + \int_s^t \Gamma_{s,u} \otimes \mathrm{d}X_u\right) = \frac{1}{2}\big(X_{s,t} \otimes \Gamma_{s,t} + \Gamma_{s,t} \otimes X_{s,t}\big).$$

We then have

$$\begin{aligned}
[\mathbf{Z}]_{s,t} &= Z_{s,t} \otimes Z_{s,t} - 2\,\mathrm{Sym}(\mathbb{Z}_{s,t}) \\
&= (X_{s,t} + \Gamma_{s,t}) \otimes (X_{s,t} + \Gamma_{s,t}) - 2\,\mathrm{Sym}(\mathbb{X}_{s,t}) - X_{s,t} \otimes \Gamma_{s,t} - \Gamma_{s,t} \otimes X_{s,t} - \Gamma_{s,t} \otimes \Gamma_{s,t} \\
&= X_{s,t} \otimes X_{s,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t}) \\
&= [\mathbf{X}]_{s,t}.
\end{aligned}$$

Part (c) Simply recall that $\mathbf{X}$ is weakly geometric if and only if $[\mathbf{X}] = 0$, and use the result of part (b).

Problem 3

Let $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ be a rough path, and suppose that $(Y, Y') \in \mathscr{D}_X^{2\alpha}$ and $(Y', Y'') \in \mathscr{D}_X^{2\alpha}$ are controlled paths. Suppose further that

$$Y_t = Y_0 + \int_0^t Y'_s \,\mathrm{d}\mathbf{X}_s + \Gamma_t$$

for all $t \in [0, T]$, for some path $\Gamma \in \mathcal{C}^{2\alpha}$. Let $f \in C^3$.

Prove that

$$f(Y_T) = f(Y_0) + \int_0^T Df(Y_u) Y'_u \,\mathrm{d}\mathbf{X}_u + \int_0^T Df(Y_u)\,\mathrm{d}\Gamma_u + \frac{1}{2}\int_0^T D^2 f(Y_u)(Y'_u \otimes Y'_u)\,\mathrm{d}[\mathbf{X}]_u.$$

Solution:

Taking a second order Taylor expansion, we have

$$f(Y_t) - f(Y_s) = Df(Y_s) Y_{s,t} + \frac{1}{2} D^2 f(Y_s)(Y_{s,t} \otimes Y_{s,t}) + O(|t-s|^{3\alpha}).$$

We have that

$$
\begin{aligned}
Y_{s,t} &= \int_s^t Y'_u \,\mathrm{d}\mathbf{X}_u + \Gamma_{s,t} \\
&= Y'_s X_{s,t} + Y''_s \mathbb{X}_{s,t} + \Gamma_{s,t} + O(|t-s|^{3\alpha}).
\end{aligned}
$$

Substituting this into the above, we obtain

$$
f(Y_t) - f(Y_s) = Df(Y_s)\big(Y'_s X_{s,t} + Y''_s \mathbb{X}_{s,t} + \Gamma_{s,t}\big) + \frac{1}{2} D^2 f(Y_s)\big(Y'_s X_{s,t} \otimes Y'_s X_{s,t}\big) + O(|t-s|^{3\alpha}).
$$

We now introduce the term

$$
D^2 f(Y_s)(Y'_s \otimes Y'_s)\mathbb{X}_{s,t} = D^2 f(Y_s)(Y'_s \otimes Y'_s)\,\mathrm{Sym}(\mathbb{X}_{s,t}) + D^2 f(Y_s)(Y'_s \otimes Y'_s)\,\mathrm{Anti}(\mathbb{X}_{s,t}),
$$

where Sym and Anti denote the symmetric and antisymmetric parts. We recall that the contraction of a symmetric tensor (here $D^2 f$) with an antisymmetric tensor (here $\mathrm{Anti}(\mathbb{X})$) always vanishes. Thus,

$$
D^2 f(Y_s)(Y'_s \otimes Y'_s)\mathbb{X}_{s,t} = D^2 f(Y_s)(Y'_s \otimes Y'_s)\,\mathrm{Sym}(\mathbb{X}_{s,t}).
$$

We then have

$$
\begin{aligned}
f(Y_t) - f(Y_s) &= Df(Y_s)\big(Y'_s X_{s,t} + Y''_s \mathbb{X}_{s,t} + \Gamma_{s,t}\big) + D^2 f(Y_s)(Y'_s \otimes Y'_s)\mathbb{X}_{s,t} \\
&\quad + \frac{1}{2} D^2 f(Y_s)\big(Y'_s X_{s,t} \otimes Y'_s X_{s,t}\big) - D^2 f(Y_s)(Y'_s \otimes Y'_s)\,\mathrm{Sym}(\mathbb{X}_{s,t}) + O(|t-s|^{3\alpha}) \\
&= Df(Y_s) Y'_s X_{s,t} + \big(Df(Y_s) Y''_s + D^2 f(Y_s)(Y'_s \otimes Y'_s)\big)\mathbb{X}_{s,t} + Df(Y_s)\Gamma_{s,t} \\
&\quad + \frac{1}{2} D^2 f(Y_s)(Y'_s \otimes Y'_s)(X_{s,t} \otimes X_{s,t} - 2\,\mathrm{Sym}(\mathbb{X}_{s,t})) + O(|t-s|^{3\alpha}) \\
&= \big(Df(Y)Y'\big)_s X_{s,t} + \big(Df(Y)Y'\big)'_s \mathbb{X}_{s,t} + Df(Y_s)\Gamma_{s,t} \\
&\quad + \frac{1}{2} D^2 f(Y_s)(Y'_s \otimes Y'_s)[\mathbf{X}]_{s,t} + O(|t-s|^{3\alpha}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f(Y_T) - f(Y_0) &= \lim_{|\pi| \to 0} \sum_{[s,t] \in \pi} \big(f(Y_t) - f(Y_s)\big) \\
&= \int_0^T Df(Y_u) Y'_u \,\mathrm{d}\mathbf{X}_u + \int_0^T Df(Y_u)\,\mathrm{d}\Gamma_u + \frac{1}{2}\int_0^T D^2 f(Y_u)(Y'_u \otimes Y'_u)[\mathbf{X}]_u.
\end{aligned}
$$

Problem 4

Suppose that $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\alpha$ and $(K, K') \in \mathscr{D}_X^{2\alpha}$ are such that the rough integral $\int_0^\cdot K_u \,\mathrm{d}\mathbf{X}_u$ takes values in $\mathbb{R}$. Let V be the path given by

$$
V_t = \exp\bigg(\int_0^t K_u \,\mathrm{d}\mathbf{X}_u - \frac{1}{2}\int_0^t (K_u \otimes K_u)\,\mathrm{d}[\mathbf{X}]_u\bigg), \qquad t \in [0,T].
$$

Prove that V is the unique solution of the rough differential equation

$$
V_t = 1 + \int_0^t V_u K_u \,\mathrm{d}\mathbf{X}_u, \qquad t \in [0,T]. \tag{1}
$$

Solution:

Define the controlled path

$$(Z, Z') := \left(\int_0^{\cdot} K_u \,\mathrm{d}\mathbf{X}_u, K \right) \in \mathscr{D}_X^{2\alpha},$$

and then let $\mathbf{Z} = (Z, \mathbb{Z})$ be the canonical rough path lift of Z as defined in the lectures, so that

$$\mathbb{Z}_{s,t} := \int_s^t Z_{s,u} \,\mathrm{d}Z_u.$$

By the associativity and consistency of rough integration, for any controlled path V, we have that

$$\int_0^t V_u K_u \,\mathrm{d}\mathbf{X}_u = \int_0^t V_u \,\mathrm{d}Z_u = \int_0^t V_u \,\mathrm{d}\mathbf{Z}_u.$$

Thus, a path V satisfies the RDE (1) if and only if it satisfies

$$V_t = 1 + \int_0^t V_u \,\mathrm{d}\mathbf{Z}_u,$$

which we recall has a unique solution. Moreover, the solution is given by

$$V_t = \exp\left(Z_t - \frac{1}{2}[\mathbf{Z}]_t \right).$$

Recalling from the lectures that

$$[\mathbf{Z}]_t = \int_0^t (K_u \otimes K_u) \,\mathrm{d}[\mathbf{X}]_u,$$

we obtain the desired representation of V.